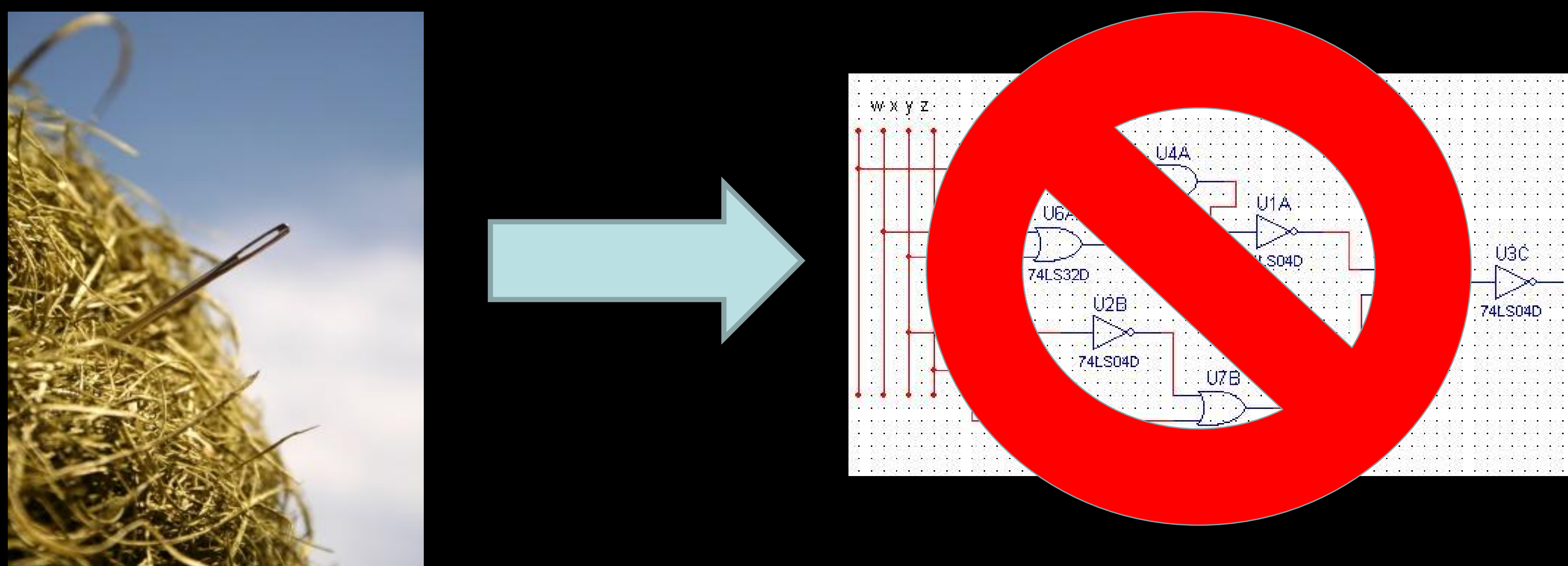
Parameterized Algorithms and Circuit Lower Bounds



Ryan Williams **Stanford**

Two Important Areas of Research

Faster FPT Algorithms for NP

Given: **Verifier V(x, y)** that reads a **k-bit** witness **y**, and runs in $(k+|x|)^{O(1)}$ time.

Find: a deterministic algorithm which

1. Runs in ***less than*** $2^k \cdot |x|^{O(1)}$ time
2. Given any input **x**, finds a **y** so that **V(x,y)** accepts (or concludes there is no **y**)

Circuit Lower Bounds

Given: Any **NP** problem (or any $\mathbf{EXP^{NP}}$ problem!)

Find: Sequence of algorithms $\{A_n\}$ such that:

1. $|A_n| \leq n^k + k$
2. On all inputs **x** of length n, $A_n(x)$ correctly solves the problem in $O(n^k)$ time.

(Alternatively, prove that no such algorithms exist!)

One May Look "Easier" Than The Other...

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Faster FPT Algorithms for NP

- **3SAT:** **$O^*(1.308^n)$ time** [H12]
- **k-Path:** **$O^*(1.66^k)$** [BHKP11]
- **Min-VC:** **$O^*(1.28^k)$** [CKX06]
  - **degree-3:** **$O^*(1.17^k)$** [M11]
- **3-Coloring:** **$O^*(1.33^n)$** [E04]
- **k-Coloring:** **$O^*(2^n)$** [BHKP08]
- **… many, many more!**

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  - degree-3: $O^*(1.17^k)$ [X10]
- 3-Coloring: $O^*(1.33^n)$ [BE05]
- Max-2-SAT: $O^*(1.8^n)$ [W05]
- … many, many more!

Circuit Lower Bounds

- We don't know how to get non-uniform algorithms that outperform these *uniform* ones

- Best lower bound known: There is a function in NP that requires circuits of size $5n + o(n)$

- It is still open whether EXP^{NP} has ***polynomial-size circuits***!

Faster Algorithms ⟹ Lower Bounds!

Faster FPT Algorithms

Deterministic algorithm for:

- CircuitSAT in $n^{O(1)} 2^k/k^{\log k}$ time (circuits with **k** inputs, **n** gates)
- FormSAT in $n^{O(1)} 2^k/k^{\log k}$ time
- ACC SAT in $n^{O(1)} 2^k/k^{\log k}$ time
- Given a circuit that's either *unsatisfiable*, or has *at least* 2^{k-1} *satisfying assignments*, determine which is the case in $n^{O(1)} 2^k/k^{\log k}$ time

(This problem is in BPP!)

Circuit Lower Bounds

Would imply:

- **NEXP $\not\subset$ P/poly** [W'10]
- **NEXP $\not\subset$ non-uniform NC^1**
- **NEXP $\not\subset$ non-uniform ACC**

[W'11]

NEXP $\not\subseteq$ P/poly

Circuit Satisfiability

Let **C** be a class of Boolean circuits

C = {Arbitrary Boolean formulas over AND and OR},
C = {Constant-depth circs}, **C** = {Arbitrary Boolean circuits}

The C-SAT Problem: Given a circuit $\mathbf{K(x_1,\ldots,x_k)} \in \mathbf{C}$ with **k inputs** and **n gates**, is there an assignment $(a_1, \ldots, a_k) \in \{0,1\}^k$ such that $\mathbf{K(a_1,\ldots,a_k) = 1}$?

C-SAT is **NP**-complete, for essentially all interesting **C**

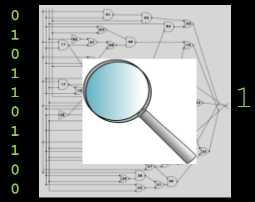
C-SAT is solvable in $2^k \cdot n^{O(1)}$ time

Connecting Algorithms + Lower Bounds

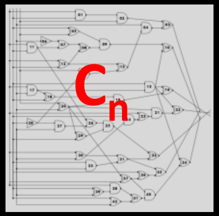
Theorem: **For many natural circuit classes C,**
IF C-SAT has a slightly faster parameterized algorithm,
THEN NEXP can't be efficiently simulated by C-circuits

Proof Plan: Assume we have two kinds of "good algorithms"

1. **Slightly faster C-SAT algorithm**



2. **Every problem in NEXP has a small C-circuit family**

$\Pi \in \mathrm{NEXP} \Rightarrow$ Π **is solved by a family** {  }

Use them to simulate every 2^n time algorithm in << 2^n time

False by the **time hierarchy theorem**!

Assume (for an appropriate circuit class **C**)

- **C-SAT with n inputs and $n^{O(1)}$ size is in $O(2^n/n^{10})$ time**
- **NEXP has polynomial-size circuits from class C**

Karp-Lipton, Meyer '80: **P = NP $\Rightarrow$ EXP $\not\subset$ P/poly**

Assume P = NP and EXP $\subset$ P/poly

EXP $\subset$ P/poly $\Rightarrow$ $\exists$ polysize circuits C encoding tableaus:

For every exponential-time machine M and every string x,

C(M,x,i,j) prints the content of the jth cell of M(x) in step i

The behavior of M(x) can be simulated in Σ_2 P :

($\exists$ C)($\forall$ i, j) [C makes consistent claims of cells j-1, j, j+1 in steps i-1, i, i+1]

coNP predicate

$\Rightarrow$ ($\exists$ C)R(x,C) , where R(x,C) is a poly-time computable predicate

NP predicate

$\Rightarrow$ M(x) is in P. But then EXP = P, contradicting the time hierarchy.

Assume (for an appropriate circuit class **C**)

- **C-SAT with n inputs and $n^{O(1)}$ size is in $O(2^n/n^{10})$ time**
- **NEXP has polynomial-size circuits from class C**

Impagliazzo-Kabanets-Wigderson '01:

NEXP $\subset$ polysize C $\Rightarrow$ $\exists$ circuit D from class C encoding tableaus:

For every ***nondeterministic*** **2^n** time machine M and every string x,

D(M,x,i,j) prints the content of the jth cell of M(x) in step i

The behavior of M(x) can be simulated in Σ_2 P :

($\exists$ D)($\forall$ i, j) [D makes consistent claims of cells j-1, j, j+1 in steps i-1, i, i+1]

Express this efficiently as an C-SAT instance??

$\Rightarrow$ ($\exists$ D)R(x,D) , where R(x,D) is an $O(2^n/n^{10})$ time predicate

$\Rightarrow$ M(x) is in *nondeterministic* $O(2^n/n^{10})$ time.

But then NTIME[2^n] $\subseteq$ NTIME[$2^n/n^{10}$],

contradicting the *nondeterministic* time hierarchy!

Definition: The Circuit Class **ACC**

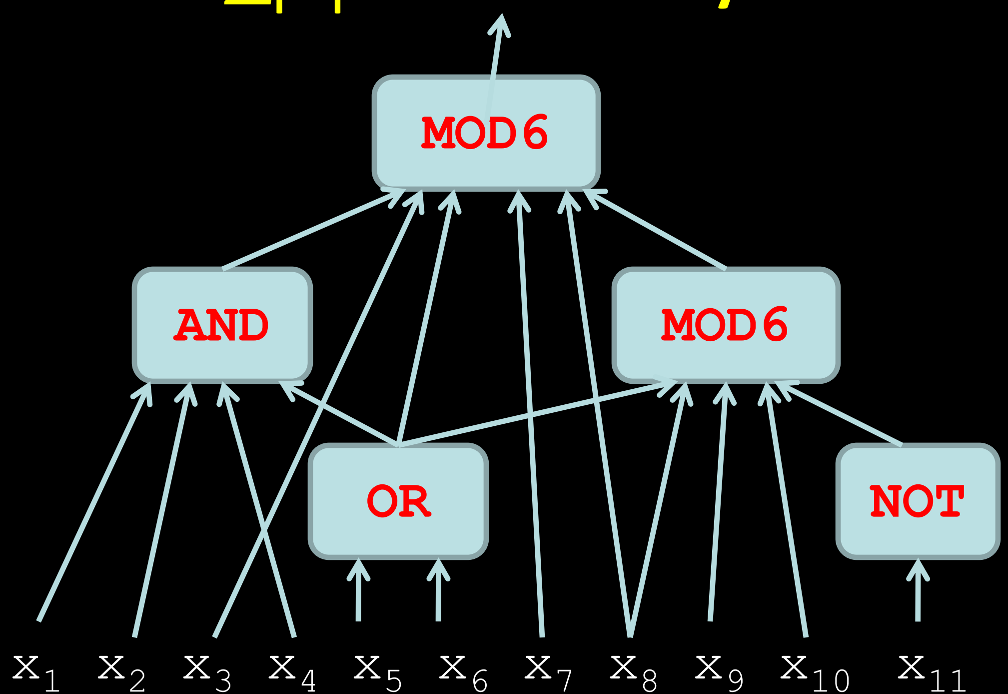
An **ACC** circuit family **{ C_n }** has the properties:

- Every C_n takes n bits of input and outputs a bit
- There is a fixed ***d*** such that every C_n has depth ***d***
- There is a fixed ***m*** such that the gates of C_n are

AND, OR, NOT, MODm (unbounded fan-in)

MODm($x_1,\ldots,x_t$) = 1 iff $\sum_i x_i$ is divisible by m

n = 11
Size = 5
Depth = 3



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Remarks

1. The default size of n^{th} circuit: **polynomial in n**
2. This is a ***non-uniform*** model of computation (Can compute some undecidable languages)
3. ACC circuits can be efficiently simulated by ***constant-layer neural networks***

Where does ACC come from?

Sipser's Program: Prove **P** $\neq$ **NP** by proving **NP** $\not\subset$ **P/poly**.
The simple combinatorial nature of circuits should make it easier to prove impossibility results.

Ajtai, Furst-Saxe-Sipser, Håstad (early 80's)
MOD2 $\notin$ **AC0** [i.e., $n^{O(1)}$ size **ACC** with ***only*** AND, OR, NOT]

Razborov, Smolensky (late 80's)
MOD3 $\notin$ (**AC0** with **MOD2** gates)
For p $\neq$ q prime, **MODp** $\notin$ (**AC0** with **MODq** gates)

Barrington (late 80's) Suggested **ACC** as the next step

Conjecture Majority $\notin$ **ACC**

No real progress since then

Proof Strategy for ACC Lower Bounds

1. **Show that faster ACC-SAT algorithms imply lower bounds against ACC**

2. **Design faster ACC-SAT algorithms!**

Theorem For all **d, m** there's an **$\varepsilon > 0$** such that **ACC-SAT** on circuits with **n** inputs, depth **d**, **MODm** gates, and **$2^{n^{\varepsilon}}$** size can be solved in **$2^{n - \Omega(n^{\varepsilon})}$ time**

Ingredients for Solving ACC SAT

Ingredients:

1. **A known representation of ACC**

   [Yao '90, Beigel-Tarui'94] Every ACC function **$f : \{0,1\}^n \to \{0,1\}$** can be expressed in the form

   $$f(x_1,\ldots,x_n) = g(h(x_1,\ldots,x_n))$$

   - **h** is a multilinear polynomial with **K** monomials and **for all $(x_1,\ldots,x_n) \in \{0,1\}^n$, $h(x_1,\ldots,x_n) \in \{0,\ldots,K\}$**
   - **K** is not "too large" *(quasipolynomial in circuit size)*
   - **$g : \{0,\ldots,K\} \to \{0,1\}$** can be an arbitrary function

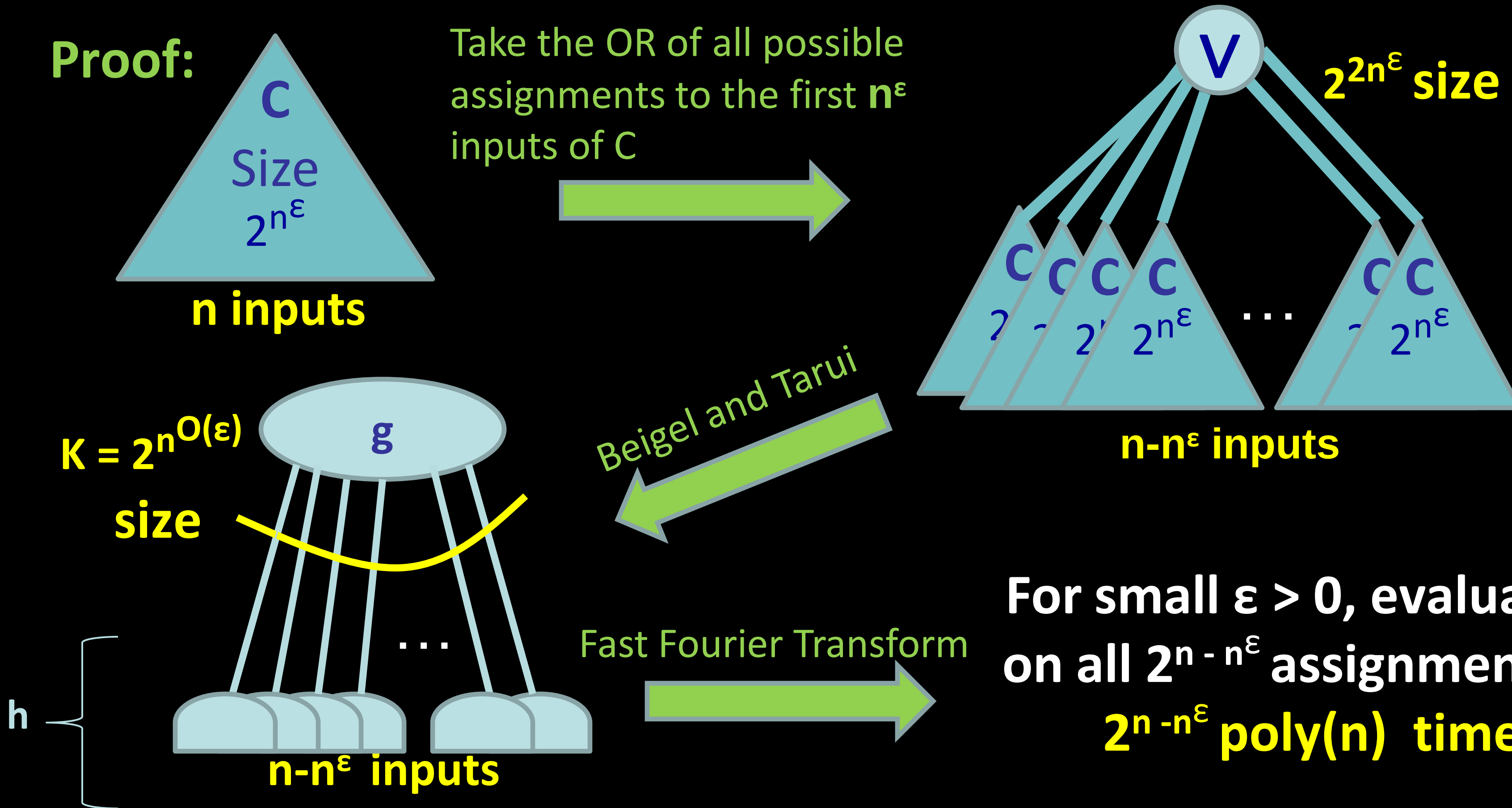
2. **"Fast Fourier Transform" for multilinear polynomials:**

   Given a multilinear polynomial h in its coefficient representation, the value h(x) can be computed over all points $x \in \{0,1\}^n$ in **2^n poly(n)** time.

ACC Satisfiability Algorithm

Theorem For all d, m there's an $\varepsilon > 0$ such that ACC[m] SAT with depth d, n inputs, $2^{n^{\varepsilon}}$ size can be solved in $2^{n - \Omega(n^{\varepsilon})}$ time

Proof:

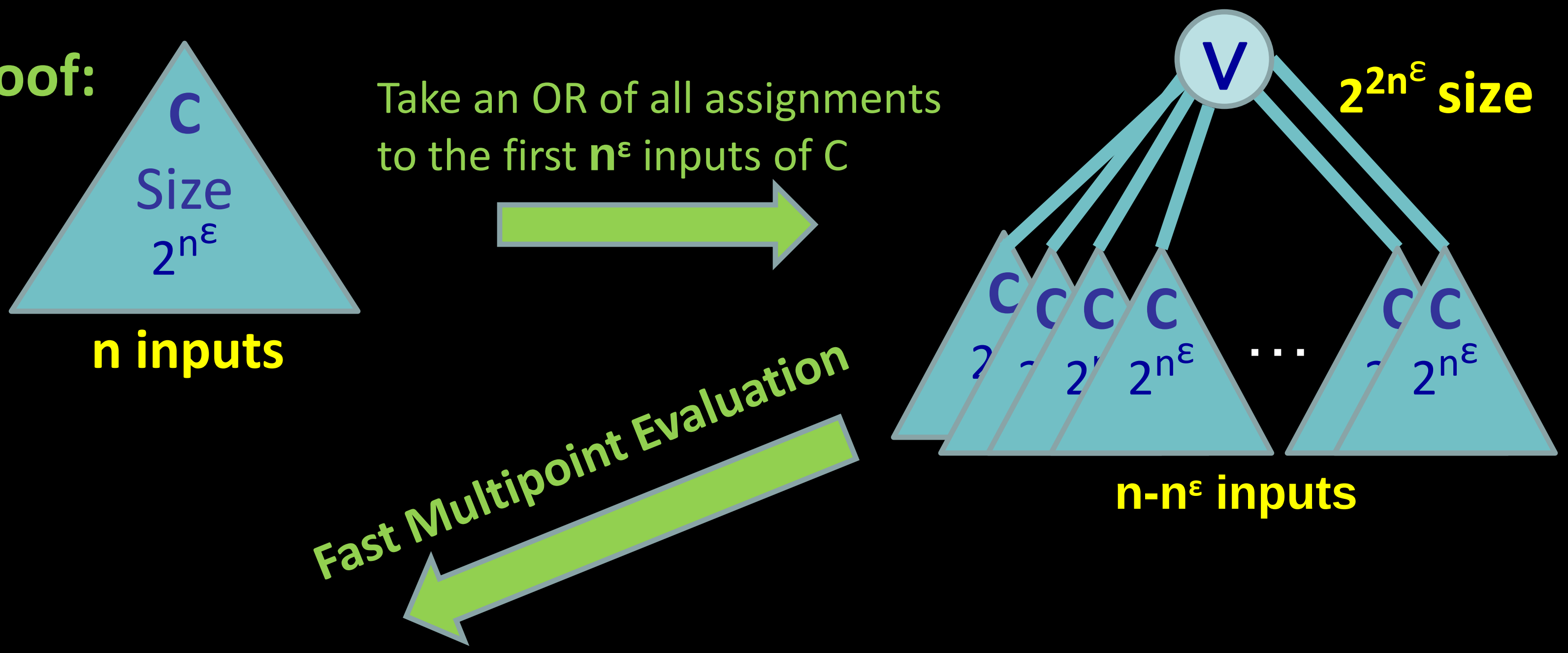


For small $\varepsilon > 0$, evaluate h on all $2^{n - n^{\varepsilon}}$ assignments in $2^{n - n^{\varepsilon}}$ poly(n) time

Fast Multipoint Circuit Evaluation Suffices for Circuit Lower Bounds!

Theorem If **Multipoint Evaluation of C-circuits of size s** can be done in 2^n poly(n) + poly(s) time, then **C-SAT** is in $o(2^n)$ time

Proof:



For small ε > 0, can evaluate circuit on all $2^{n - n^\varepsilon}$ assignments in $2^{n - n^\varepsilon}$ poly(n) + poly(2^{2n^ε}) time

Future Work

- **Replace NEXP with simpler complexity classes**
  Very recently: replaced with (NEXP ∩ coNEXP)
  For EXP: may need to improve on exhaustive search for more complex problems
- **Replace ACC with stronger circuits**
  Design SAT algorithms for other circuit classes!
  Using strong versions of PCP Theorem: very weak derandomization suffices
- **Find more connections between algorithms and lower bounds!**

Thank you!