Parameterized complexity in computational game theory

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Introduction

Definition

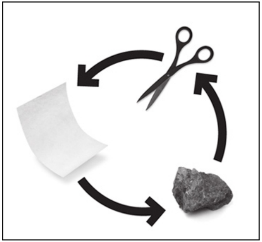
Game theory

The mathematical study of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants.

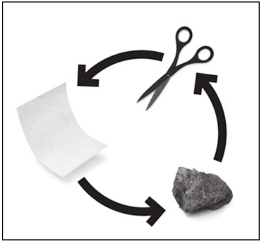
Applications:

- Politics.
- Behavioural sciences.
- Economics.
- Internet.

Rock Paper Scissors

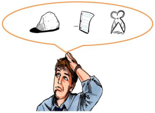


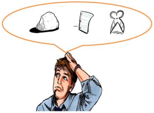
Rock Paper Scissors



- Rock beats Scissors,
  Rock 1 $\lhd\rhd$ Scissors -1.
- Scissors beats Paper,
  Scissors 1 $\lhd\rhd$ Paper -1.
- Paper beats Rock,
  Paper 1 $\lhd\rhd$ Rock -1.
- If the two objects are the same, then they both get 0.

Rock Paper Scissors (Cont.)

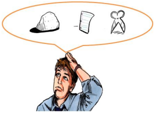


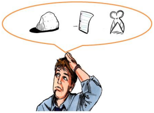


	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Table: The payoff matrix of players in Rock, Paper, Scissors game.

Rock Paper Scissors (Cont.)





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What is the best way to play?

Game Theory vs Algorithmic Game Theory

Game theory provides solution concepts such as:

- Mixed Nash equilibria
- Pure Nash equilibria
- Correlated equilibria
- Dominant strategies
- etc.

While, algorithmic game theory aims to compute the solutions proposed by the game theory.

Prisoner's Dilemma– A classical example

Two men that are charged with a crime and held in separate cells.

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- if both confess, then each will be jailed for four years.
- However, both prisoners know that if neither confesses, then they will just be jailed for two years as the prosecution would have insufficient evidence for a more severe penalty.

Prisoners' Dilemma

	Confess	Not Confess
Confess		
Not Confess		

Prisoners' Dilemma

	Confess	Not Confess
Confess	4,4	
Not Confess		

Prisoners' Dilemma

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Not Confess	5,1	

Prisoners' Dilemma

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Confess	4,4	1,5
Not Confess	5,1	2,2

Prisoners' Dilemma

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What is the best decision for each prisoner?

Nash equilibrium (NE)

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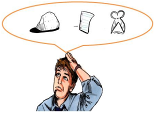
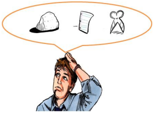
The strategy (Confess, Confess) is a pure Nash equilibrium.

Properties of pure Nash equilibrium

- Pure Nash equilibria are easy to compute (in bi-matrix games).
- Existence of pure Nash equilibria are not guaranteed.
  - Rock Paper Scissors.





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Mixed strategy

- **Mixed strategy**: A probability distribution (collection of weights) which corresponds to how frequently each move is to be played.

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Example

Playing Rock 50% of the time and Scissors 50% of the time;

- Is there any equilibrium point with this type of strategies (mixed strategies) ?

Nash Existence Theorem

Every finite n-player game has an equilibrium point [4].

- The proof is based on Brouwer fixed point theorem.
- **Brouwer theorem**: Every continuous function f from a convex and compact set to itself has a fixed point ($f(x) = x$).
- It is well-known that Brouwer fixed point theorem does not have a constructive proof.
- Nash's proof does not provide an algorithm for the computation of equilibria.

How hard is it to compute Nash equilibria?

- Computing one Nash equilibrium for k-player games is unlikely to be solved in polynomial time [3, 1, 2].
- Most decision problems regarding computation of Nash equilibria are **NP**-Complete.
  - A Nash equilibrium that each player uses at least k moves with positive probability.
  - A Nash equilibrium that all the strategies played with nonzero probability by a player are played with the same probability.

Coping with hard problems

- Approximation algorithm.
- Heuristics.
- Brute-force search.
- Parametrization.

Parameterizing the Nash decision problems!

Parameterized results

k-Minimal Nash support

Finding Nash equilibrium with smallest support

- **Support of mixed strategy**: The set of moves that have positive probabilities is called the support of mixed strategy.

Example

Playing Rock 50% of the time and Scissors 50% of the time and never Paper;
Support = {Rock, Scissors}

- Why the support of mixed strategies are important?
  - If the support of Nash strategies is known, then the computation of Nash equilibrium is determined by solving a system of inequalities.
- The size of support would be an important parameter.
- Nash equilibria with small support are interesting.

k-Minimal Nash support (MNS)

Instance : A game $\mathcal{G}$=(A,B).
Parameter : Positive integer k.
Question : Does $\mathcal{G}$ have a Nash equilibrium $(\vec{x}, \vec{y})$ such that $\max\{\|supp(\vec{x})\|, \|supp(\vec{y})\|\} \leq k$?

Theorem

k-Minimal Nash support *is $W[2]$-hard.*

Set Cover

Instance : A family $S = \{S_1, ..., S_r\}$ of r subsets of set $N = \{1, ..., n\}$ that covers N, that is $\bigcup_{i=1,...,r} S_i = N$.
Parameter : Positive integer $k \leq r$.
Question : Does S have a subset of size at most k such that it covers N?

Theorem

Set Cover *is* W*[2]-Complete.*

Construct a parameterized reduction

$$\textsc{Set Cover} \Longrightarrow k\text{-}\textsc{Minimal Nash support}$$

$$(N, S, r, k) \longmapsto (A_{(r+1)\times(n+1)}, -A_{(r+1)\times(n+1)}, k)$$

- $r+1$ is the number of strategies of first player,
- $n+1$ is the number of strategies for the second player, and
- $A_{(r+1)\times(n+1)}$ is defined by the following function:

$$a_{ij} = \begin{cases} 1 & \text{if } i \leq r,\ j \leq n,\ j \in S_i, \\ 0 & \text{if } i \leq r,\ i \leq n,\ j \notin S_i, \\ 1/k & \text{if } i \leq r,\ j = n+1, \\ 1/2r & \text{if } i = r+1. \end{cases}$$

Instance of reduction

- $N = \{1, 2, 3, 4, 5\}$, $S = \{S_1, S_2, S_3, S_4\}$.
- $k = 2$, $r = 4$, $n = 5$.
- $S_1 = \{1, 3\}$, $S_2 = \{2, 4, 5\}$, $S_3 = \{1, 2\}$, $S_4 = \{3, 4, 5\}$.

	1	2	3	4	5	6
S_1	1	0	1	0	0	1/2
S_2	0	1	0	1	1	1/2
S_3	1	1	1	0	0	1/2
S_4	0	0	1	1	1	1/2
S_5^*	1/8	1/8	1/8	1/8	1/8	1/8

It is a parameterized reduction!

Theorem

The cover S of the set N has a sub-cover of size k or less if and only if the game $\mathcal{G}=(A,-A)$ has a Nash equilibrium such that the size of the support of the Nash strategy is at most k.

k-uniform Nash

Uniform Nash equilibria

- A mixed strategy $\vec{x}$ is called *uniform*, if for every i,j in the support of $\vec{x}$, we have $x_i = x_j$.
- Probably, it is the simplest way to combine the pure strategies.
- In game Rock, Paper, Scissors the following mixed strategies constitute a uniform Nash equilibrium.

$$\begin{aligned} x &= (1/3, 1/3, 1/3) \\ y &= (1/3, 1/3, 1/3) \end{aligned}$$

k-Uniform Nash

Instance : A game $\mathcal{G}=(A,B)$.
Parameter : Positive integer k.
Question : Is there a uniform Nash equilibrium $(\vec{x},\vec{x})$ such that $\|supp(\vec{x})\| = k$.

Theorem

k-Uniform Nash *is* $W[2]$*-Complete.*

MAX CLIQUE

Instance : Graph $G{=}(V,E)$ and positive integer k.
Parameter : Positive integer k.
Question : Is there a subset V' of V such that $G_{V'}$ constitutes a maximal clique of size k?.

- Special MAX CLIQUE: The given graph has a vertex that connects to all other vertices (We call it vertex 1).

Theorem

MAX CLIQUE *is* $W[2]$*-Complete.*

Reduction

$$\textsc{Max Clique} \Longrightarrow k\text{-}\textsc{Uniform Nash}$$

$$(G, k) \longmapsto ((I, M), k)$$

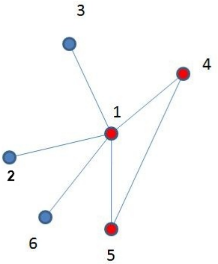
- The number of strategies for both player is n, the number of vertices of graph G, we label them with $\{1, \ldots, n\}$.
- The payoff matrix of first player, $I = [a_{ij}]$, and the payoff matrix of the second player, $M = [m_{ij}]$ is defined by the following functions.

$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad m_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

This is a parameterized reduction!

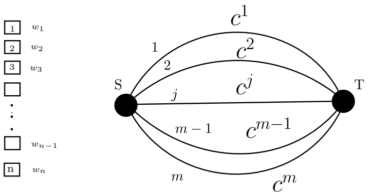
Theorem

$(\vec{x},\vec{x})$ is a k-uniform Nash equilibrium of the game (I,M), if and only if the subgraph induced by support of strategy $\vec{x}$ constitutes a maximal clique of size k.



		1/3	0	0	1/3	1/3	0
		1	2	3	4	5	6
1/3	1	0	1	1	1	1	1
0	2	1	0	0	0	0	0
0	3	1	0	0	0	0	0
1/3	4	1	0	0	0	1	0
1/3	5	1	0	0	0	0	0
0	6	1	0	0	0	0	0

Routing Games

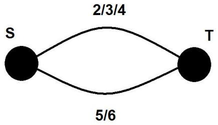


- A source node S and a terminal node T.
- A set of m parallel links from S to T.
- A capacity c^j for each link $j \in \{1, 2, \ldots, m\}$.
- A set $N = \{1, 2, \ldots, n\}$, of n users
- A vector of traffic weights, w_1,w_2,. . . , w_n, where the i-th user has traffic $w_i > 0$.

Definitions– routing games

- Identical links: Equal capacity to all the links.
- A pure strategy for a user i is a link j in $\{1, 2, \ldots, m\}$.
- A pure strategy profile is an n-tuples $(l_1, l_2, \ldots l_n)$, when user i chooses link l_i in $\{1, 2, \ldots, m\}$
- The cost for a user i is the cost of link it chooses.
- Every routing game admits at least one pure Nash equilibrium.

Example



- The cost of players that uses top link (2+3+5=9).
- The cost of players that uses the bottom link (5+6=11).
- The social cost is equal $\max\{9, 11\} = 11$.

Finding Nash equilibria with best social optima

- The individual (non-cooperative) optimization of utility does not lead to a social optimal outcome.
- The price of stability of a game is the ratio between the best objective function value of a Nash equilibrium of the game and the optimal outcome.

Definition

The makespan of a strategy profile $P = (l_1, l_2, \ldots, l_n)$ *is defined as:*

$$C_{max}(P) = \max_{i \in \{1,2,\ldots,n\}} C_i(P).$$

BEST NASH EQUILIBRIUM

Instance : A routing game $\mathcal{G}$ with identical links.
Parameter : $k \in \mathbb{N}$.
Question : Is there a pure Nash equilibrium P with $C_{max}(P) \leq k$?

Theorem

BEST NASH EQUILIBRIUM *is in FPT.*

Proof.

By a reduction to INTEGER LINEAR PROGRAMMING. □

INTEGER LINEAR PROGRAMMING

Instance : A matrix $A \in \mathbb{Z}^{m \times k}$ and a vector $\vec{b} \in \mathbb{Z}^m$.
Parameter: $k \in \mathbb{N}$.
Question : Does there exist a non-negative integral vector $\vec{x}$ such that $A \cdot \vec{x} \leq \vec{b}$?

Theorem

INTEGER LINEAR PROGRAMMING *is in FPT.*

Proof

- Let (I,k) be an instance of Best Nash Equilibrium.
- If there exists a user i, where $w(i) > k$, then I is a NO-instance.

Rule 1:

- For every user i, $w(i) \leq k$

Pattern: An assignment of a k-tuple $\bar{a}_j = (a_{j,1}, a_{j,2}, \ldots, a_{j,k})$ to a link j, where $a_{j,i} \in \{0, 1, \ldots, k\}$.

- A pattern $\bar{a}_j$ indicates that, exactly $a_{j,i}$ users with traffic i placed their traffic to link $j \in \{1, 2, \ldots, m\}$ (for $i \in \{1, 2, \ldots, k\}$).

Cont. proof

Idea:

- Introduce integer variables that indicate, for a given pattern $\bar{a}$, how many links share the same $\bar{a}$ pattern.

Plus:

1- The cost of each pattern should not exceed the parameter k.

2- The solution must be a Nash equilibrium.

Cont. proof Cond. 1

Cond 1:

– We just consider patterns with cost of size at most k.

Cost: Cost of a pattern $\bar{a} = (a_1, a_2, \ldots, a_k)$ is defined as

$$C(\bar{a}) = \sum_{i=1}^{k} i \cdot a_i$$

.

– Let $\mathcal{C} = \{\bar{a} : C(\bar{a}) \leq k\}$.

– The cardinality of $\mathcal{C}$ is bounded by $(k+1)^k$.

Notion of Dominance:

- $\bar{a} = (a_1, a_2, \ldots, a_k)$, $\bar{b} = (b_1, b_2, \ldots, b_k) \in \mathcal{C}$
- We say that $\bar{a}$ is dominated by $\bar{b}$ ($\bar{a} \preceq \bar{b}$), if

$$\exists i \in \{1, 2, \ldots, k\} \text{ with } a_i \neq 0 \text{ and } c(\bar{b}) + i < c(\bar{a}).$$

Means:

- If $\bar{a} \preceq \bar{b}$, there is a user with traffic i in a link who has an incentive to move to another link.

Solution:

- We can describe a solution by determining for each pattern $\bar{a}$, how many links follow pattern $\bar{a}$.

Const.

- No two different patterns dominate each other.

Defn.

- For every pattern $\bar{a} \in \mathcal{C}$, we introduce an integer variable $x_{\bar{a}}$, where the variable $x_{\bar{a}}$ denotes the number of links that follow the pattern $\bar{a}$.
- Let b_j be the number of users $i \in \{1, 2, \ldots, n\}$ with traffic $w_i = j$, for $j \in \{1, 2, , \ldots, k\}$.

Integer quadratic program

(1) $\sum_{\bar{a} \in \mathcal{C}} x_{\bar{a}} = m,$
(2) $\forall i \quad (1 \leq i \leq k \Rightarrow \sum_{\bar{a}=(a_1,a_2,\ldots,a_k)} a_i \cdot x_{\bar{a}} = b_i),$
(3) $\forall \bar{a}, \bar{b} \in \mathcal{C} \quad (\bar{a} \preceq \bar{b} \Rightarrow x_{\bar{a}} \cdot x_{\bar{b}} = 0),$
(4) $\forall \bar{a} \in \mathcal{C} \quad x_{\bar{a}} \in \mathbb{N} \cup \{0\}.$

Solution:

- Equation (1) ensures that each link is assigned only one pattern.
- Equation (2) ensure that we assign the right number of users for each value of traffic $j \in \{1, 2, \ldots, k\}$.
- Equations (3) should be satisfied, since it encodes the dominance constraint.

Equation (3) transforms to linear equations

- We can replace each equation $x_{\bar{a}} \cdot x_{\bar{b}} = 0$ with a combination of two linear inequalities by introducing a new integer variable $z_{ab} \in \{0, 1\}$ as follows:

$$x_{\bar{a}} \cdot x_{\bar{b}} = 0 \quad \equiv \quad x_{\bar{a}} \leq m \cdot z_{ab} \text{ and } x_{\bar{b}} \leq m \cdot (1 - z_{ab}) \; .$$

- The problem reduces to the integer linear program.

Theorem

BEST NASH EQUILIBRIUM *is in FPT.*

Some FPT Results

1) In sparse symmetric imitation win-lose games a sample Nash equilibrium can be found in FPT-time.

Imitation

- A game that the payoff matrix of the first player is identity matrix.

Sparse

- The payoff matrix of the second player has at most k non-zero entries in row and each column.

2) Given a subsets of strategies of each player, we can determine in FPT-time whether there exist a Nash equilibrium that the supports of Nash strategies are the subset of the given sets.

The road ahead

- Determine the parameterized complexity of best Nash equilibria when:
  - The links are not identical.
  - Underlying graph are not parallel links.
  - ...
- Determine the parameterized complexity of worst Nash equilibria.
- ...

Thanks

Questions?

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