

Parameterized complexity in computational game theory

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1 Introduction

- What is game theory?
- Nash equilibrium

2 Complexity of computation of Nash equilibria

3 Nash equilibria with smallest support

4 Uniform Nash equilibria

5 Best Nash equilibria in congestion games

6 More results

Introduction

Definition

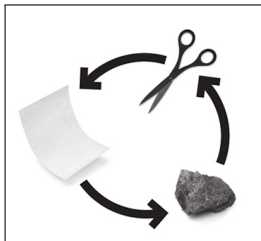
Game theory

The mathematical study of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants.

Applications:

- Politics.
- Behavioural sciences.
- Economics.
- Internet.

Rock Paper Scissors



Rock Paper Scissors



- Rock beats Scissors, Rock 1 $\triangleleft\triangleright$ Scissors -1.
- Scissors beats Paper, Scissors 1 $\triangleleft\triangleright$ Paper -1.
- Paper beats Rock, Paper 1 $\triangleleft\triangleright$ Rock -1.
- If the two objects are the same, then they both get 0.

Rock Paper Scissors (Cont.)



	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Table: The payoff matrix of players in Rock, Paper, Scissors game.

Rock Paper Scissors (Cont.)



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What is the best way to play?

Game Theory vs Algorithmic Game Theory

Game theory provides solution concepts such as:

- Mixed Nash equilibria
- Pure Nash equilibria
- Correlated equilibria
- Dominant strategies
- etc.

While, algorithmic game theory aims to compute the solutions proposed by the game theory.

Prisoner's Dilemma— A classical example

Two men that are charged with a crime and held in separate cells.

Prisoner's Dilemma— A classical example

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- If one confesses and the other does not, the confessor will be awarded a smaller sentence (1 year jail), and the other will be jailed for five years,

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 - if both confess, then each will be jailed for four years.

Prisoner's Dilemma— A classical example

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- If one confesses and the other does not, the confessor will be awarded a smaller sentence (1 year jail), and the other will be jailed for five years,
 - if both confess, then each will be jailed for four years.
 - However, both prisoners know that if neither confesses, then they will just be jailed for two years as the prosecution would have insufficient evidence for a more severe penalty.

└ Introduction

└ What is game theory?

Prisoners' Dilemma



	Confess	Not Confess
Confess		
Not Confess		

└ Introduction

└ What is game theory?

Prisoners' Dilemma



	Confess	Not Confess
Confess	4,4	
Not Confess		

└ Introduction

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Prisoners' Dilemma



	Confess	Not Confess
Confess	4,4	1,5
Not Confess		

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Prisoners' Dilemma



	Confess	Not Confess
Confess	4,4	1,5
Not Confess	5,1	

└ Introduction

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Prisoners' Dilemma



	Confess	Not Confess
Confess	4,4	1,5
Not Confess	5,1	2,2

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Prisoners' Dilemma



	Confess	Not Confess
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What is the best decision for each prisoner?

Nash equilibrium (NE)

An equilibrium point represents a stable state such that in that situation no player has incentive to deviate from the equilibrium points, when the opponents' strategies become known.

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The strategy (Confess, Confess) is a pure Nash equilibrium.

Properties of pure Nash equilibrium

- Pure Nash equilibria are easy to compute (in bi-matrix games).
- Existence of pure Nash equilibria are not guaranteed.
 - Rock Paper Scissors.



	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Mixed strategy

- **Mixed strategy:** A probability distribution (collection of weights) which corresponds to how frequently each move is to be played.

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Example

Playing Rock 50% of the time and Scissors 50% of the time;

- Is there any equilibrium point with this type of strategies (mixed strategies) ?

Nash Existence Theorem

Every finite n -player game has an equilibrium point [4].

- The proof is based on Brouwer fixed point theorem.
- **Brouwer theorem:** Every continuous function f from a convex and compact set to itself has a fixed point ($f(x) = x$).
- It is well-known that Brouwer fixed point theorem does not have a constructive proof.
- Nash's proof does not provide an algorithm for the computation of equilibria.

How hard is it to compute Nash equilibria?

- Computing one Nash equilibrium for k -player games is unlikely to be solved in polynomial time [3, 1, 2].
- Most decision problems regarding computation of Nash equilibria are **NP**-Complete.
 - A Nash equilibrium that each player uses at least k moves with positive probability.
 - A Nash equilibrium that all the strategies played with nonzero probability by a player are played with the same probability.

Coping with hard problems

- Approximation algorithm.
- Heuristics.
- Brute-force search.
- **Parametrization.**

Parameterizing the Nash decision problems!

Parameterized results

└ Nash equilibria with smallest support

k -MINIMAL NASH SUPPORT

Finding Nash equilibrium with smallest support

- **Support of mixed strategy:** The set of moves that have positive probabilities is called the support of mixed strategy.

Example

Playing Rock 50% of the time and Scissors 50% of the time and never Paper;

Support = {Rock, Scissors}

- Why the support of mixed strategies are important?
 - If the support of Nash strategies is known, then the computation of Nash equilibrium is determined by solving a system of inequalities.
- The size of support would be an important parameter.
- Nash equilibria with small support are interesting.

└ Nash equilibria with smallest support

k -MINIMAL NASH SUPPORT (MNS)

Instance : A game $\mathcal{G}=(A,B)$.

Parameter : Positive integer k .

Question : Does \mathcal{G} have a Nash equilibrium (\vec{x}, \vec{y}) such that $\max\{\|supp(\vec{x})\|, \|supp(\vec{y})\|\} \leq k$?

Theorem

k -MINIMAL NASH SUPPORT is $W[2]$ -hard.

SET COVER

- Instance :** A family $S = \{S_1, \dots, S_r\}$ of r subsets of set $N = \{1, \dots, n\}$ that covers N , that is $\bigcup_{i=1, \dots, r} S_i = N$.
- Parameter :** Positive integer $k \leq r$.
- Question :** Does S have a subset of size at most k such that it covers N ?

Theorem

SET COVER is $W[2]$ -Complete.

Construct a parameterized reduction

SET COVER \implies k -MINIMAL NASH SUPPORT

$$(N, S, r, k) \longmapsto (A_{(r+1) \times (n+1)}, -A_{(r+1) \times (n+1)}, k)$$

- $r + 1$ is the number of strategies of first player,
- $n + 1$ is the number of strategies for the second player, and
- $A_{(r+1) \times (n+1)}$ is defined by the following function:

$$a_{ij} = \begin{cases} 1 & \text{if } i \leq r, j \leq n, j \in S_i, \\ 0 & \text{if } i \leq r, i \leq n, j \notin S_i, \\ 1/k & \text{if } i \leq r, j = n + 1, \\ 1/2r & \text{if } i = r + 1. \end{cases}$$

└ Nash equilibria with smallest support

Instance of reduction

- $N = \{1, 2, 3, 4, 5\}$, $S = \{S_1, S_2, S_3, S_4\}$.
- $k = 2$, $r = 4$, $n = 5$.
- $S_1 = \{1, 3\}$, $S_2 = \{2, 4, 5\}$, $S_3 = \{1, 2\}$, $S_4 = \{3, 4, 5\}$.

	1	2	3	4	5	6
S_1	1	0	1	0	0	1/2
S_2	0	1	0	1	1	1/2
S_3	1	1	1	0	0	1/2
S_4	0	0	1	1	1	1/2
S_5^*	1/8	1/8	1/8	1/8	1/8	1/8

It is a parameterized reduction!

Theorem

The cover S of the set N has a sub-cover of size k or less if and only if the game $\mathcal{G}=(A,-A)$ has a Nash equilibrium such that the size of the support of the Nash strategy is at most k .

k -uniform Nash

Uniform Nash equilibria

- A mixed strategy \vec{x} is called *uniform*, if for every i, j in the support of \vec{x} , we have $x_i = x_j$.
- Probably, it is the simplest way to combine the pure strategies.
- In game Rock, Paper, Scissors the following mixed strategies constitute a uniform Nash equilibrium.

$$x = (1/3, 1/3, 1/3)$$

$$y = (1/3, 1/3, 1/3)$$

k -UNIFORM NASH

Instance : A game $\mathcal{G}=(A,B)$.

Parameter : Positive integer k .

Question : Is there a uniform Nash equilibrium (\vec{x}, \vec{x}) such that $\|supp(\vec{x})\| = k$.

Theorem

k -UNIFORM NASH is $W[2]$ -Complete.

MAX CLIQUE

Instance : Graph $G=(V,E)$ and positive integer k .

Parameter : Positive integer k .

Question : Is there a subset V' of V such that $G_{V'}$ constitutes a maximal clique of size k ?

- **Special MAX CLIQUE:** The given graph has a vertex that connects to all other vertices (We call it vertex 1).

Theorem

MAX CLIQUE is $W[2]$ -Complete.

Reduction

MAX CLIQUE \implies k -UNIFORM NASH

$$(G, k) \longmapsto ((I, M), k)$$

- The number of strategies for both player is n , the number of vertices of graph G , we label them with $\{1, \dots, n\}$.
- The payoff matrix of first player, $I = [a_{ij}]$, and the payoff matrix of the second player, $M = [m_{ij}]$ is defined by the following functions.

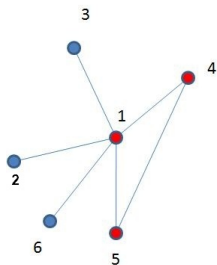
$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

$$m_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

This is a parameterized reduction!

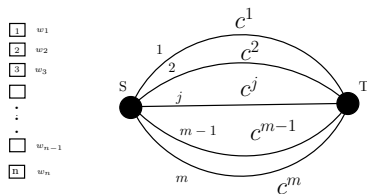
Theorem

(\vec{x}, \vec{x}) is a k -uniform Nash equilibrium of the game (I, M) , if and only if the subgraph induced by support of strategy \vec{x} constitutes a maximal clique of size k .



		1/3	0	0	1/3	1/3	0
		1	2	3	4	5	6
1/3	1	0	1	1	1	1	1
0	2	1	0	0	0	0	0
0	3	1	0	0	0	0	0
1/3	4	1	0	0	0	1	0
1/3	5	1	0	0	0	0	0
0	6	1	0	0	0	0	0

Routing Games

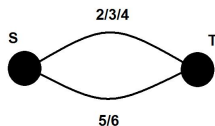


- A source node S and a terminal node T .
- A set of m parallel links from S to T .
- A capacity c^j for each link $j \in \{1, 2, \dots, m\}$.
- A set $N = \{1, 2, \dots, n\}$, of n users
- A vector of traffic weights, w_1, w_2, \dots, w_n , where the i -th user has traffic $w_i > 0$.

Definitions– routing games

- Identical links: Equal capacity to all the links.
- A pure strategy for a user i is a link j in $\{1, 2, \dots, m\}$.
- A pure strategy profile is an n -tuples (l_1, l_2, \dots, l_n) , when user i chooses link l_i in $\{1, 2, \dots, m\}$
- The cost for a user i is the cost of link it chooses.
- Every routing game admits at least one pure Nash equilibrium.

Example



- The cost of players that uses top link ($2+3+5=9$).
- The cost of players that uses the bottom link ($5+6=11$).
- The social cost is equal $\max\{9, 11\} = 11$.

Finding Nash equilibria with best social optima

- The individual (non-cooperative) optimization of utility does not lead to a social optimal outcome.
- The price of stability of a game is the ratio between the best objective function value of a Nash equilibrium of the game and the optimal outcome.

Definition

The makespan of a strategy profile $P = (l_1, l_2, \dots, l_n)$ is defined as:

$$C_{max}(P) = \max_{i \in \{1, 2, \dots, n\}} C_i(P).$$

BEST NASH EQUILIBRIUM

Instance : A routing game \mathcal{G} with identical links.

Parameter : $k \in \mathbb{N}$.

Question : Is there a pure Nash equilibrium P with $C_{max}(P) \leq k$?

Theorem

BEST NASH EQUILIBRIUM *is in FPT*.

Proof.

By a reduction to INTEGER LINEAR PROGRAMMING. □

INTEGER LINEAR PROGRAMMING

Instance : A matrix $A \in \mathbb{Z}^{m \times k}$ and a vector $\vec{b} \in \mathbb{Z}^m$.

Parameter: $k \in \mathbb{N}$.

Question : Does there exist a non-negative integral vector \vec{x} such that $A \cdot \vec{x} \leq \vec{b}$?

Theorem

INTEGER LINEAR PROGRAMMING *is in FPT*.

Proof

- Let (I, k) be an instance of BEST NASH EQUILIBRIUM.
- If there exists a user i , where $w(i) > k$, then I is a NO-instance.

Rule 1:

- For every user i , $w(i) \leq k$

Pattern: An assignment of a k -tuple $\bar{a}_j = (a_{j,1}, a_{j,2}, \dots, a_{j,k})$ to a link j , where $a_{j,i} \in \{0, 1, \dots, k\}$.

- A pattern \bar{a}_j indicates that, exactly $a_{j,i}$ users with traffic i placed their traffic to link $j \in \{1, 2, \dots, m\}$ (for $i \in \{1, 2, \dots, k\}$).

Cont. proof

Idea:

- Introduce integer variables that indicate, for a given pattern \bar{a} , how many links share the same \bar{a} pattern.

Plus:

- 1- The cost of each pattern should not exceed the parameter k .
- 2- The solution must be a Nash equilibrium.

Cont. proof Cond. 1

Cond 1:

- We just consider patterns with cost of size at most k .

Cost: Cost of a pattern $\bar{a} = (a_1, a_2, \dots, a_k)$ is defined as

$$C(\bar{a}) = \sum_{i=1}^k i \cdot a_i$$

- Let $\mathcal{C} = \{\bar{a} : C(\bar{a}) \leq k\}$.
- The cardinality of \mathcal{C} is bounded by $(k + 1)^k$.

Notion of Dominance:

- $\bar{a} = (a_1, a_2, \dots, a_k), \bar{b} = (b_1, b_2, \dots, b_k) \in \mathcal{C}$
- We say that \bar{a} is dominated by \bar{b} ($\bar{a} \preceq \bar{b}$), if

$$\exists i \in \{1, 2, \dots, k\} \text{ with } a_i \neq 0 \text{ and } c(\bar{b}) + i < c(\bar{a}).$$

Means:

- If $\bar{a} \preceq \bar{b}$, there is a user with traffic i in a link who has an incentive to move to another link.

Solution:

- We can describe a solution by determining for each pattern \bar{a} , how many links follow pattern \bar{a} .

Const.

- No two different patterns dominate each other.

Defn.

- For every pattern $\bar{a} \in \mathcal{C}$, we introduce an integer variable $x_{\bar{a}}$, where the variable $x_{\bar{a}}$ denotes the number of links that follow the pattern \bar{a} .
- Let b_j be the number of users $i \in \{1, 2, \dots, n\}$ with traffic $w_i = j$, for $j \in \{1, 2, \dots, k\}$.

Integer quadratic program

- (1) $\sum_{\bar{a} \in \mathcal{C}} x_{\bar{a}} = m,$
- (2) $\forall i \quad (1 \leq i \leq k \Rightarrow \sum_{\bar{a}=(a_1, a_2, \dots, a_k)} a_i \cdot x_{\bar{a}} = b_i),$
- (3) $\forall \bar{a}, \bar{b} \in \mathcal{C} \quad (\bar{a} \preceq \bar{b} \Rightarrow x_{\bar{a}} \cdot x_{\bar{b}} = 0),$
- (4) $\forall \bar{a} \in \mathcal{C} \quad x_{\bar{a}} \in \mathbb{N} \cup \{0\}.$

Solution:

- Equation (1) ensures that each link is assigned only one pattern.
- Equation (2) ensure that we assign the right number of users for each value of traffic $j \in \{1, 2, \dots, k\}.$
- Equations (3) should be satisfied, since it encodes the dominance constraint.

Equation (3) transforms to linear equations

- We can replace each equation $x_{\bar{a}} \cdot x_{\bar{b}} = 0$ with a combination of two linear inequalities by introducing a new integer variable $z_{ab} \in \{0, 1\}$ as follows:

$$x_{\bar{a}} \cdot x_{\bar{b}} = 0 \equiv x_{\bar{a}} \leq m \cdot z_{ab} \text{ and } x_{\bar{b}} \leq m \cdot (1 - z_{ab}) .$$

- The problem reduces to the integer linear program.

Theorem

BEST NASH EQUILIBRIUM *is in FPT.*

Some FPT Results

- 1) In sparse symmetric imitation win-lose games a sample Nash equilibrium can be found in FPT-time.

Imitation

- A game that the payoff matrix of the first player is identity matrix.

Sparse

- The payoff matrix of the second player has at most k non-zero entries in row and each column.
- 2) Given a subsets of strategies of each player, we can determine in FPT-time whether there exist a Nash equilibrium that the supports of Nash strategies are the subset of the given sets.

The road ahead

- Determine the parameterized complexity of best Nash equilibria when:
 - The links are not identical.
 - Underlying graph are not parallel links.
 - ...
- Determine the parameterized complexity of worst Nash equilibria.
- ...

Thanks

Questions?

[More results](#)

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