Adaptive Analysis of Algorithms

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Sorting is a core problem

Central to the debate about models of computation

- comparison-based vs sorting integers
- worst-case vs expected case (maybe best case)
- lower bounds and optimality ($O()$, $\Omega()$, $\Theta()$).
- problems vs algorithm
- internal memory vs external memory
- parallel vs sequential

Sorting is a core problem (cont)

An ideal case to initiate students on the analysis and design of algorithms

- (and data structures).
- theoretical and experimental algorithmics
- algorithmic engineering (Quicksort / Insertion Sort)

A focus on the instances

A-Sort [3] seems to be the origin of the notion of 'adaptive' [2].

- Verifying an input sequence is sorted is $\Theta(n)$ time.
- Sorting (comparison-based) is $\Theta(n \log n)$.
- Both statements can be seen as remarks about the expected case (just the distribution of instances is extreme).

Should not need to do as much work if there is only a bit of disorder to remove.

A bi-dimensional (multi-dimensional) view on algorithm complexity

Adaptive algorithm

- (originally not a view on problem complexity)
- the complexity of the algorithm is a smoothly growing function
  - of a measure of instance-hardness (disorder)
  - the size of the input

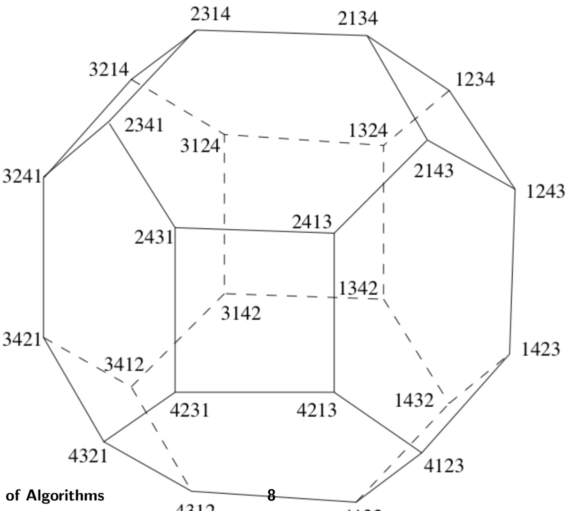
Inversions

- number of inversions.
- Let $Inv(\pi) = Inv(\pi, Id)$ (or Kendall-Tau) [distance, measure of disorder, measure of pre-sortedness, right-invariant metric $Inv(\pi, \sigma) = Inv(\pi \circ \tau, \sigma \circ \tau)$]

$$Inv(X = \langle x_1, x_2, \ldots x_n \rangle) = \|(i,j)| i < j \text{ and } x_i > x_j\|$$

Minimum number of adjacent swaps to bring the sequence into sorted order.

Illustration



Insertion Sort

STRAIGHT INSERTION SORT (the insertion data structure is an array)

- $Inv(x) + n - 1$ comparisons
- $Inv(x) + 2n - 1$ data moves

Improve the data structure (just place a finger and count only comparisons)

$$n \log(1 + Inv(X)/n).$$

Lower bounds

- $below(z, n, M) = \{X \in S_n \mid |X| = n \text{ and } M(X) \leq z\}$
- in the comparison-based model of computation the comparison tree has height at least $\Omega(\log \|below(z, n, M)\|)$.

Optimal adaptivity in the worst-case

$$T_s(X) \in O(\max\{|X|, \log \|below(z, n, M)\|).$$

Instance easiness can be measured in many ways

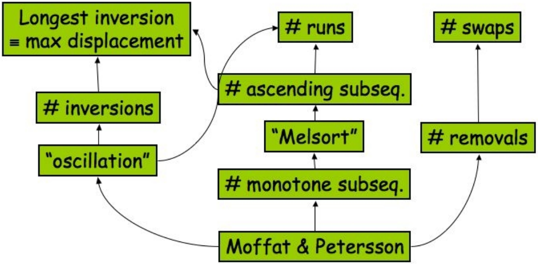
Operational

- *Exchanges* (swaps) – minimum number of exchanges to bring the sequence into sorted order.
- *Rem* – minimum number of removals to eave something sorted
- *Runs* (step downs) — passes for external sort

Hierarchy of measures of disorder

M_1 is algorithmicly finer than M_2 if and only if
whenever A is optimal adaptive with respect to M_1, then it is also optimally adaptive with respect to M_2.

Illustration



Where things were left

- Optimal algorithm (comparisons) for finest measure of disorder [Moffat & Petersson]
- Does there exists a minimal element for the hierarchy ?
- Does there exist an optimal algorithm for the optimum?
  - Iacono 2001, Bădoiu & Demain 2004, Bădoiu, Cole, Demaine, Iacono 2006.

Adaptivity —Expected case

- Makes perfect sense for randomized algorithms
- Expectation [required resources] (time/space) is a smoothly growing function of the instance easiness.

Adaptive Analysis

$\|X\|$ vs $M(X)$	0	1	2	...	k
1					
2					
3					
4					
...					
n			$f(n,k)$		

Objectives

- $f(n,k)$ monotonicly increasing for each fixed n
- proportional to
  $below(z,n,M) = \{X \in P \mid |X| = n \text{ and } M(X) \leq k\}$

Parameterized Complexity

$\|X\|$ vs k	0	1	2	$\dots$	k
1					
2					
3					
4					
$\dots$					
n		$pol(n)f(k)$			

Objectives

- Understand the frontier of hardness
- avenue to break intractability

Adaptivity in NP Problems

- $below(z, n, M) = \{X \in P \mid |X| = n \text{ and } M(X) \leq z\}$
- very close notion to parameterized complexity
- z is the parameter, M is the function of instance easiness (does this lead to the next chapter in parameterized complexity?)
- recall the argument about hierarchies of measures

Contrast between adaptive algorithms and parameterized algorithms

VERTEX COVER

- Let $G=(V,E)$, and we measure instance easiness as

$$\sum_{\text{Connected Component}\, C} \sum_{i=1}^{n-1} i \cdot \sharp \text{ vertices of degree i}$$

Adaptivity vs parameterization

- Notion of measure of instance easiness (could be the parameter)
- The maximum value of the measure is $k << n$.
- Adaptivity seems to make more sense in the class P.
- Adaptive makes sense for any resource (time, number of processors, space, number of messages) proportional to instance easiness.

Adaptivity vs parameterization

Illustration

- Measures of structural simplicity
- Tree-With (how tree like), Path-width (how Path-like), genus (how planar like).
- "Decision" version vs "Optimization" version
- Tricks also used in the adaptive case (because computing the measure may be as hard as solving the problem).
  1. for a scheme $k = 0, \ldots, \max\{M(X)\}$ Apply algorithm for $M(X) = k$.
  2. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two algorithms, respectively optimal with respect to measures of easiness M_1 and M_2, then an algorithm that runs them alternatively is optimum with respect to both measures.

Distance-based and Representative-based Clustering

- Given X set of n points (vectors $\vec{x}_i \in \Re^d$) find a partition $C_1, C_2, \ldots C_k$ ($\bigcup C_i = X$) that minimizes the loss (error).
- Total square error: Find representatives $\vec{c}_1, \ldots \vec{c}_k$ such that

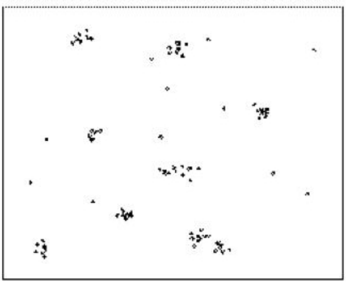
$$\sum_{j=1}^{k} \sum_{\vec{x} \in C_j} dist(\vec{x}, rep[C_j])^2$$

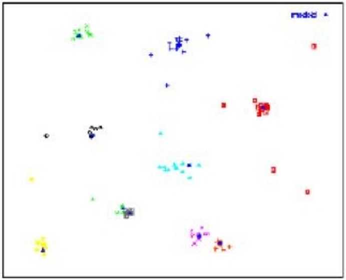
- Total error: Find representatives $\vec{c}_1, \ldots \vec{c}_k$ such that

$$\sum_{j=1}^{k} \sum_{\vec{x} \in C_j} dist(\vec{x}, rep[C_j])$$

- Medoids (discrete case): $\vec{x}_i \in X$.

Illustration





Geometric difference of criteria

A consensus problem

The case $k = 1$ and dist=Euclid.

- Total square error:

$$\sum_{\vec{x} \in C_j} dist(\vec{x}, rep[C_j])^2$$

  - Solution is center of mass (Minimizes distortion).
- Total error:

$$\sum_{\vec{x} \in C_j} dist(\vec{x}, rep[C_j])$$

  - Fermat-Webber Problem (Geometric Median).
  - No solution by digital computers.
- Discrete case: $C \subset X$
  - Problem is in XP (Test all subsets of size k).

Status of clustering from adaptive analysis

Clusterability

- Notions of how easy (instance easiness) is to cluster a particular instance X in k clusters [1]
  1. Center-perturbation clusterability.
  2. Worst-pair-ratio clusterability.
  3. Separability clusterability.
  4. Variance-ration clusterability.
  5. Clusterability to a target cluster.
- Type A Results: Clusterability for one notion may not mean clusterability for the other.
- Type B Results: If an instance has high clusterability for one measure, then it is "polynomial" to find a "good" clustering.
- Type C Results: Computing "clusterability" is NP-Hard

Center-perturbation clusterability

Center-based clustering

- Centers (and thus clusterings) $\{\vec{c}_1, \vec{2}, \ldots, \vec{c}_k\}$ are ϵ-close to $\{\vec{c}'_1, \vec{c}'_2, \ldots, \vec{c}'_k\}$ if $\forall j \|\vec{c}_j - \vec{c}'_j\| \leq \epsilon$.
- X is $(\epsilon,\ \delta,\ k)$-clusterable (for $k \geq 1$ and $\epsilon, \delta \geq 0$) if $\forall C$ a center-based clustering of X that is ϵ-close to some optimal clustering

$$\mathcal{L}(C) \leq (1+\delta) OPT_{\mathcal{L},k}(X).$$

Illustration:

1. $\mathcal{L}(C) = \sum_{j=1}^{k} \sum_{\vec{x} \in C_j} Euclid(\vec{x}, rep[C_j])^2$.
2. $OPT_{\mathcal{L},k}(X) = \min\{\mathcal{L}(C) \mid C \text{ is clustering of } X\}$.

Type B Result:

If X is $(rad(X)/\sqrt(l), \delta)$-center perturbation clusterable, then there is an algorithm that runs in polynomial time in n and outputs a cluster C so that

$$\mathcal{L}(C) \leq (1+\delta) OPT_{\mathcal{L},k}(X).$$

- Complexity is actually $O(n^{lk})$, i.e. polynomial only for fixed k (and fixed l).
- $rad(X)$ is the radius of the minimum sphere that contains X.

Algorithm:

1. $C_A \leftarrow$; $L \leftarrow$ all k tuples with entries an l-sequence of elements of X. `/*` A sample with replacement of kl elements from X `*/`
2. `for` each element of L:
   2.1 find the center of mass c_j of each l-sequence
   2.2 find the clustering $\hat{C}$ induced by the c_j's (Voronoi partition)
   2.3 `if` $C_A =$ or $\mathcal{L}(\hat{C}) < \mathcal{L}(C_A)$, `then` $C_A \leftarrow \hat{C}$.
3. `return` C_A

Worst-pair-ratio clusterability

- For clustering C of X,

$$split(C) = \min\{dist(\vec{x},\vec{y}) \mid \vec{x} \in C_i, \vec{y} \in C_j, i \neq j\}$$

$$width(C) = \max\{dist(\vec{x},\vec{y}) \mid \vec{x} \in C_i, \vec{y} \in C_i\}$$

- "Cluster-quality" of a clustering C with respect to X

$$WPR(C,X) = \frac{split(C)}{width(C)}.$$

- WPR_k clusterability

$$WPR_k(X) = \max\{WPR(C,X) \mid C \text{ is } k \text{ clustering of } X\}.$$

Type B Result:

If $WPR_k(X) \geq 1$ for some $k > 2$, we can find a k-clustering C with maximum split over width ration in $O(n^2 \log n)$ time where $n = |X|$.

1. Algorithms is single-linkage clustering until k components.
2. Correctness: If there is a clustering C (with k non-trivial clusters!) such that $width(C) < split(C)$, then there is only one such clustering.

Separability clusterability

Drop in loss relative to number k of clusters.

- The k-means loss

$$\mathcal{L}_k(C,X) = \sum_{j=1}^{k} \sum_{\vec{x} \in C_j} Euclid(\vec{x}, rep[C_j])^2.$$

- The set X is (k,ϵ)-separable if

$$OPT_{C \text{ is } k \text{ clustering}}[\mathcal{L}_k(C,X)] \leq \epsilon \, OPT_{C' \text{ is } k-1 \text{ clustering}}[\mathcal{L}_{k-1}(C',X)]$$

- The separability $S_k(X) \in [0,1)$ is $\inf\{\epsilon > 0 \mid X \text{ is } \epsilon-\text{separable}\}$ (smaller value, easier to cluster).

Type B Result:

If X is $(2,\epsilon^2)$-separable, then a 2-clustering with k-means loss

$$\mathcal{L}_k(C,X) \leq \frac{OPT_{C \text{ is 2 clustering}}[\mathcal{L}_2(C,X)]}{(1-\rho)}$$

can be found with probability $1 - O(\rho)$ in time $O(dn)$ where $\rho = \Theta(\epsilon^2)$.

1. Approximation algorithm.
2. Probabilistic algorithm.
3. Theoretical algorithm.

Variance-ratio Clusterability

- Variance of $X = \sigma^2(X) = \frac{1}{\|X\|}\sum_{\vec{x}\in X}\|\vec{x} - \text{mean}(X)\|^2$.
- k-clustering $C = \{X_1, X_2, \ldots, X_k\}$, proportion $p_i = \|X_i\|/\|X\|$.
- Between cluster variance

$$B_C(X) = \sum_{j=1}^{k} p_i\|\text{mean}(X_i) - \text{mean}(X)\|^2.$$

- Within cluster variance $W_C(X) = \sum_{j=1}^{k} p_i\sigma^2(X_i)$.
- Variance-Ratio Clusterability

$$VR_k(X) = \max_{C \text{ is a } k \text{ clustering}} \frac{B_C(X)}{W_C(X)}.$$

Type B Result:

Observations:

- $\sigma^2(X) = W_C(X) + B_C(X)$.
- $nW_C(X) = k$-means loss $= \mathcal{L}_k(C, X)$.

Therefore, $VR_2(X) = \frac{1}{S_2(X)} - 1$ for all X.

- Equivalence of measures of clusterability for $k = 2$.
- Algorithms for separability also work for variance-ratio.

Status of clustering from parameterized complexity

More common in the context of a graph $G(V, E)$.

- Sometimes weights for edges $w(e)$ with $e \in E$.
- r-DOMINATING SET
- Is there a set $C \subset V$ of size k ($\|C\| = k$) so that $\forall v \in V$ there is $c \in C$ so that $dist(v, c) < r$.
- Vanilla DOMINATING SET ($w(e) = 1$, $\forall e$ and $r = 1$) is unlikely to be FPT;
- but FPT for special cases (planar).
- However, few implementations.

Connexion between adaptivity and parameterized complexity

Determine the complexity:

- INSTANCE: A set X of n vectors with "measure" of clusterability k.
- PARAMETER: k.
- QUESTION: Does X have a clustering of "quality" k.

Investigate combinations of "measures" and "quality" (or is the problem trivial).
Produce adaptive algorithms (optimization version).

Specific Open Problem

In the context of a graph $G(V, E)$.

- Sometimes weights for edges $w(e)$ with $e \in E$.
- r-DOMINATING SET
- Is there a set $C \subset V$ of size k ($\|C\| = k$) so that $\forall v \in V$ there is $c \in C$ so that $dist(v, c) < r$.
- In the special case the the clusterability is high (for example, the worst-pair-ratio larger than 1 implies polynomial time).
- FPT? where the parameter k is (inversely) related to the clusterability (Conjecture: FPT for instances with clusterability larger than $1/k$).
- Deliver good implementations.

Adaptivity, does it matter?

- For sorting, the additional machinery usually causes too much overhead.
- Closest to best engineered approach (carefully engineer QUICKSORT until instances are small enough, then apply a pass of INSERTION SORT).
- Eradicate BUBBLE SORT

Hierarchicly finest measure

- For measures of disorder
- For competitive algorithms (Lopez-Ortiz discussion on LRU)
  - need suitable model of optimality
- For other environments of adaptive algorithms
  - Shortest Path (Dijkstra)

Other problems

Erik Demaine and several others

- Searching
- Sets
- Curves
- Integrals

Finding $M(X)$

- Are there problems where finding (approximating) $M(X)$ can be done much faster than actually solving P.
- A slight variation like the local-search parameterized complexity that should have sense for NP-Complete problems.

Thanks

Question?

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