Is Ryser's formula for the Permanent optimal?

Holger Dell

Dániel Marx

Thore Husfeldt

Nina Taslaman

Martin Wahlén

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(we mean uniform formulas here)

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3-SAT : Satisfiability of Boolean 3-CNF formulas

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Sparsification Lemma. [Impagliazzo, Paturi, Zane '01]
ETH $\Leftrightarrow$ k-SAT is not computable in time $2^{o(\#\text{clauses})}$.

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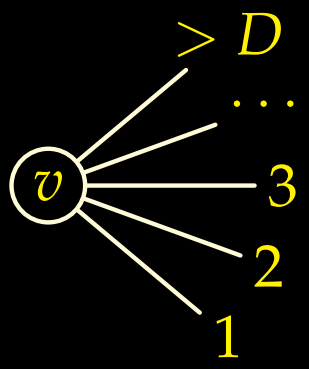

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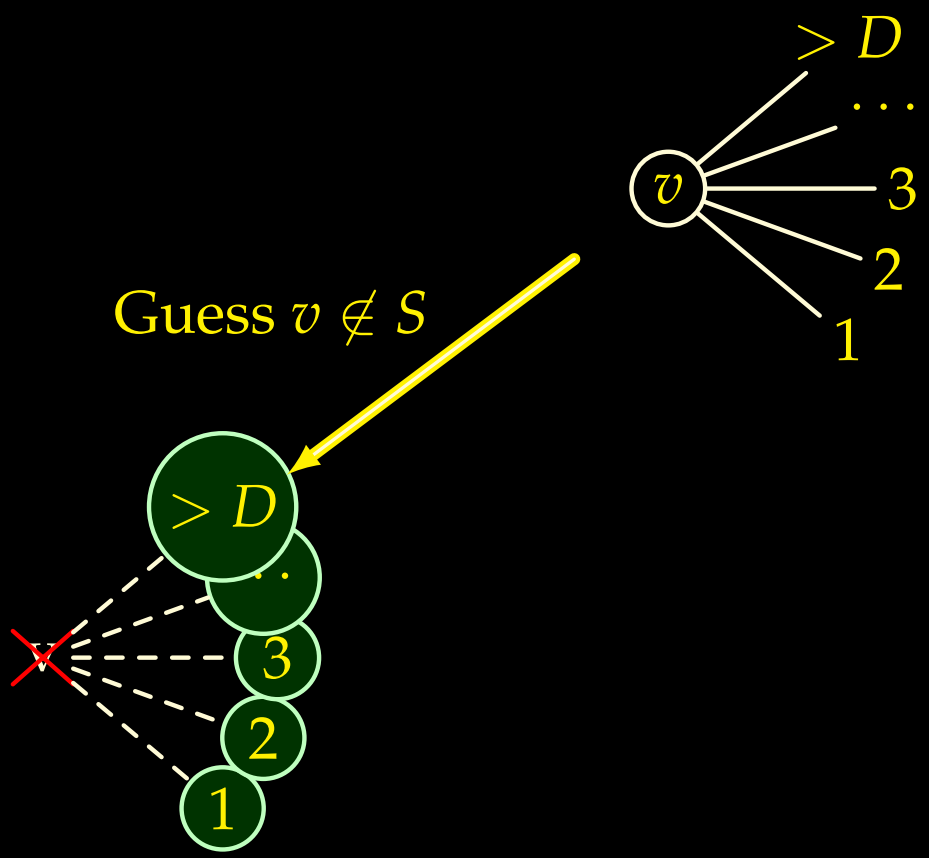

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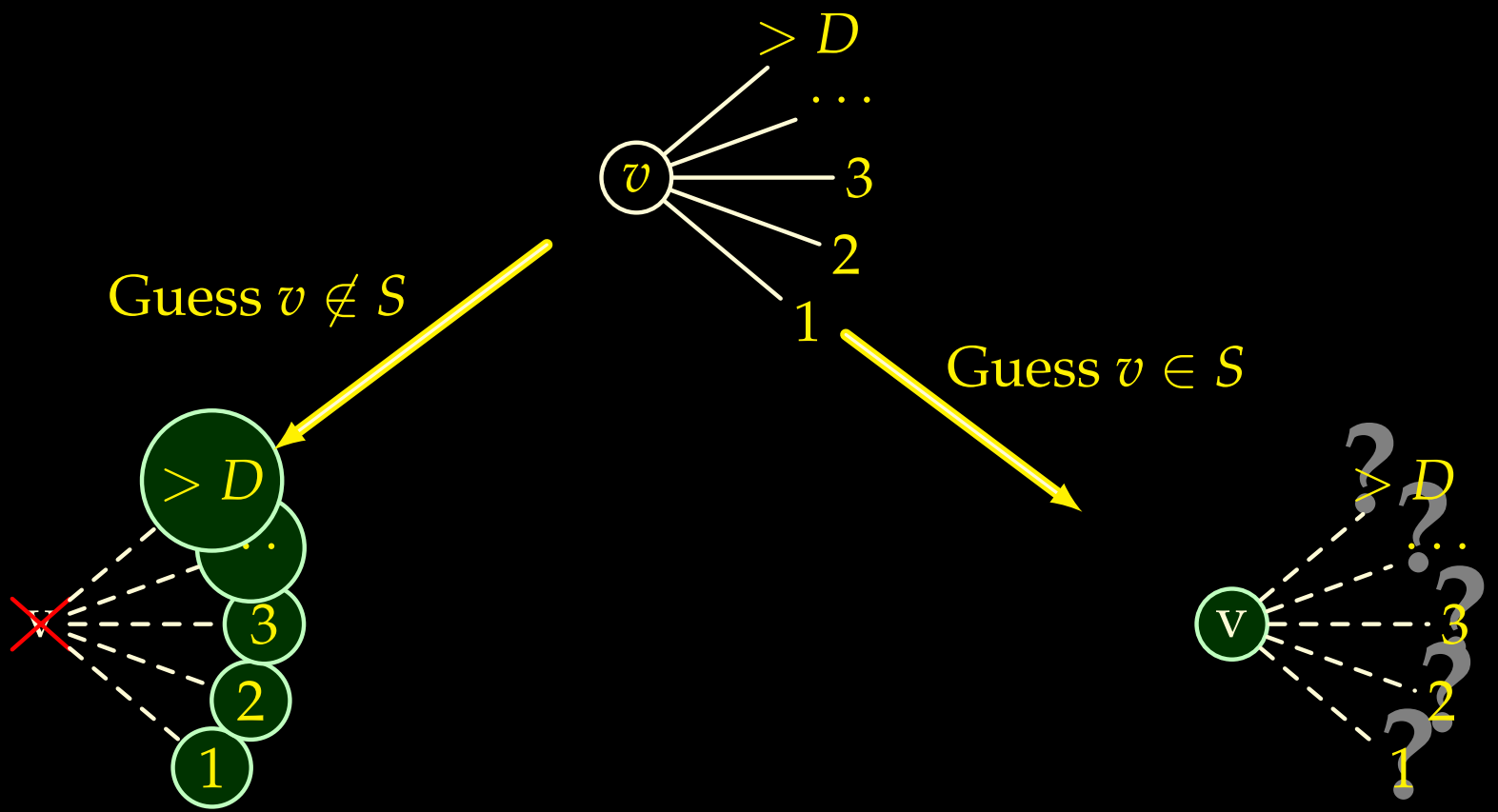

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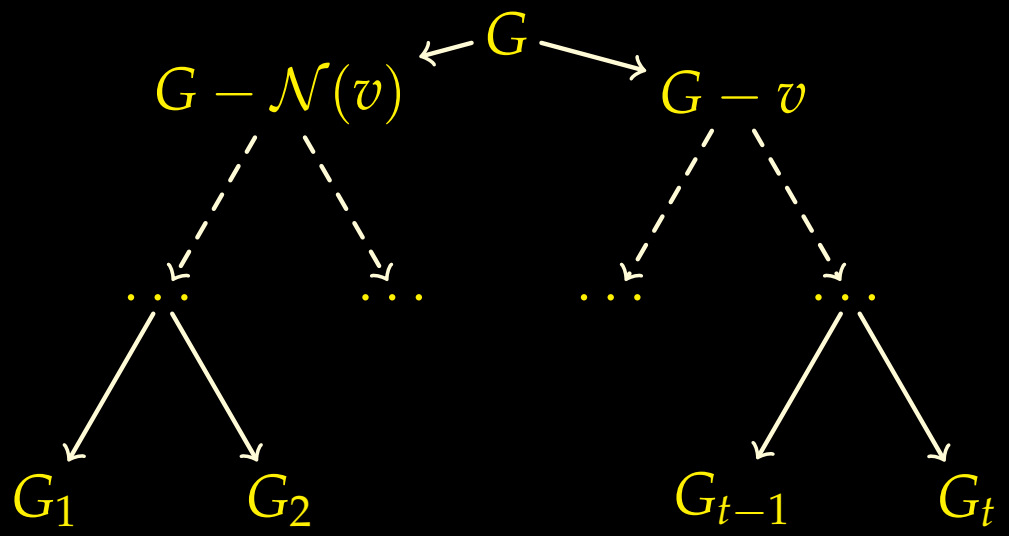

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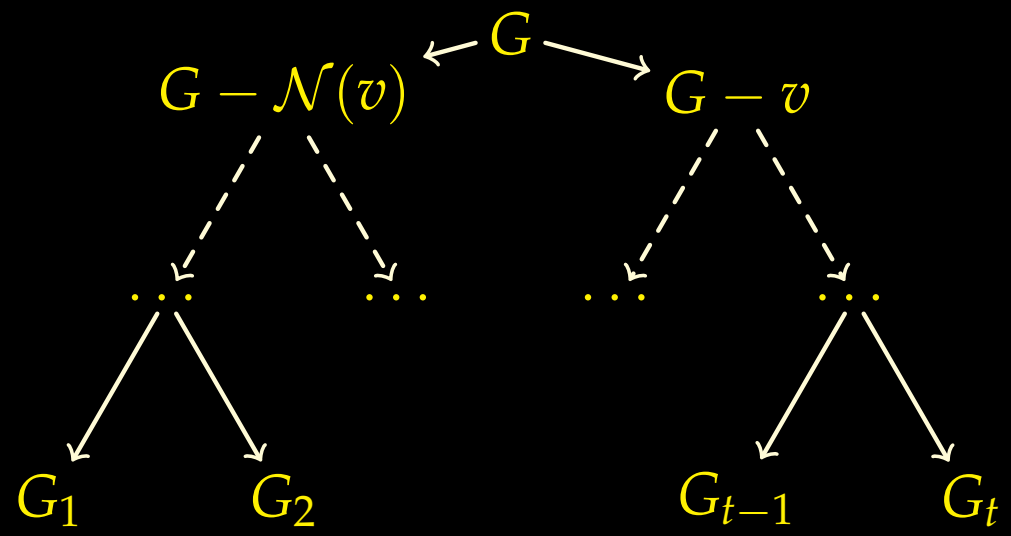

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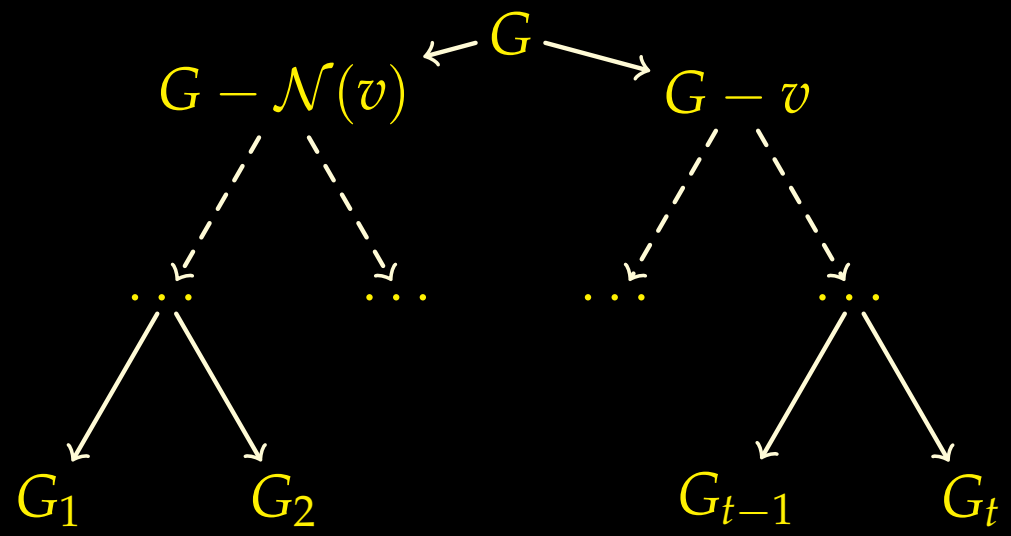

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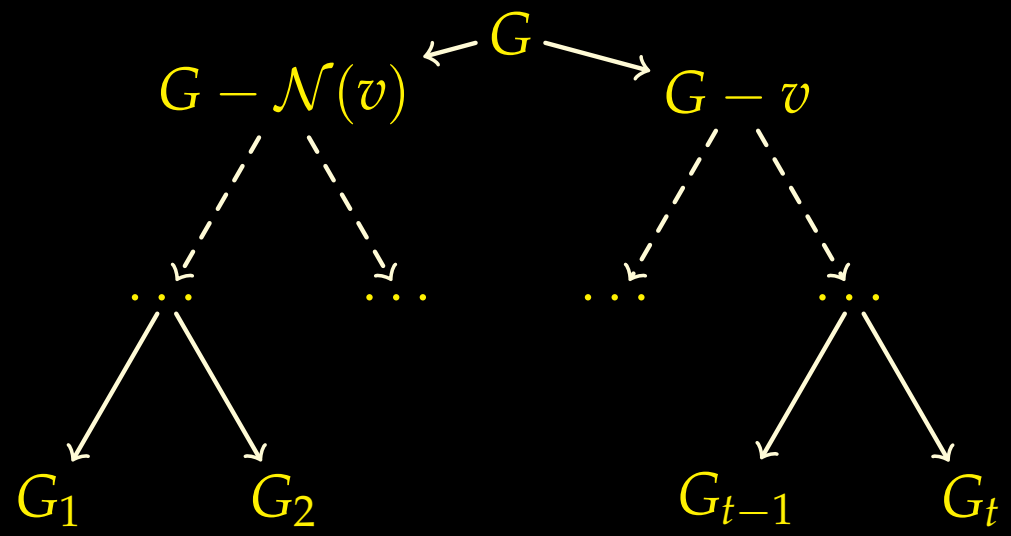

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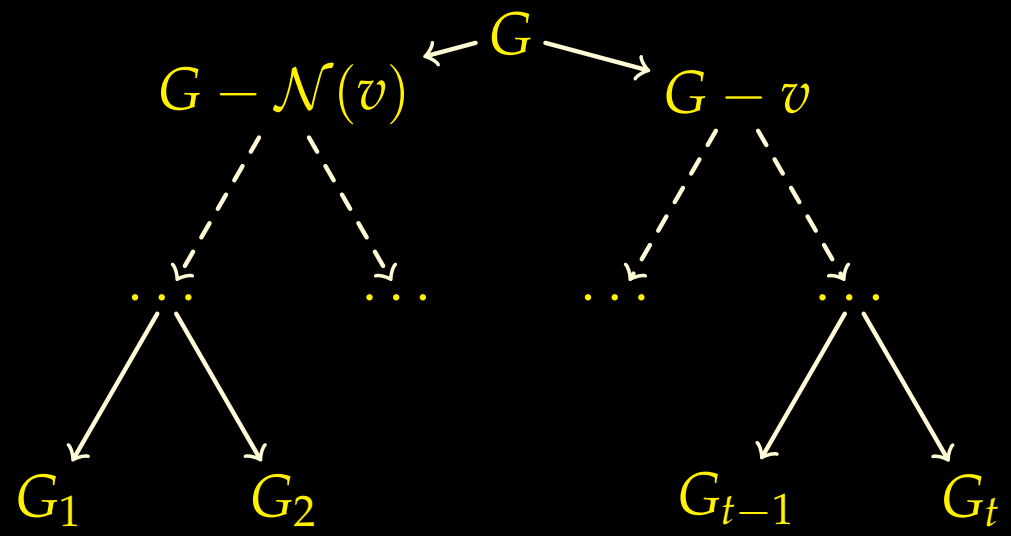

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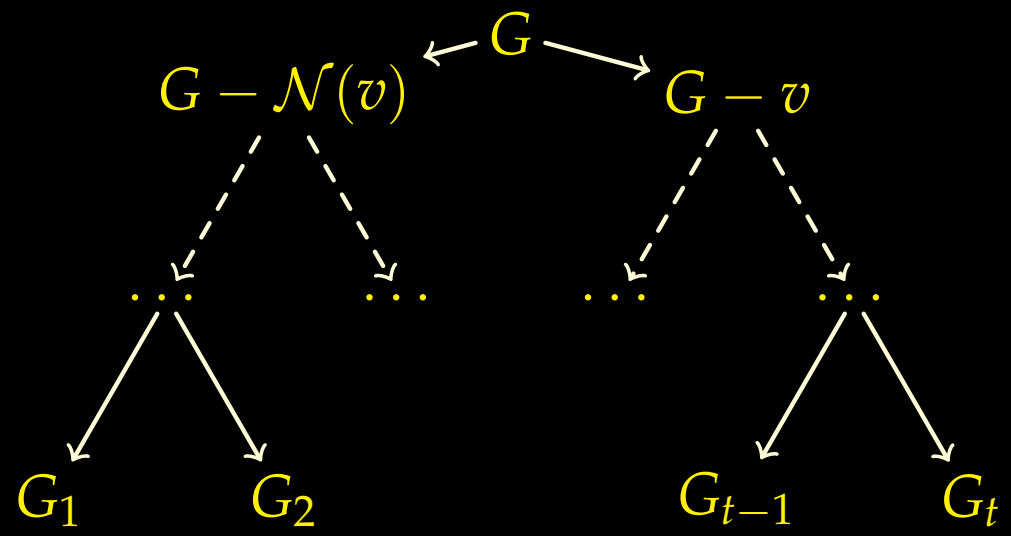

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Theorem. [Impagliazzo, Paturi, Zane '01]
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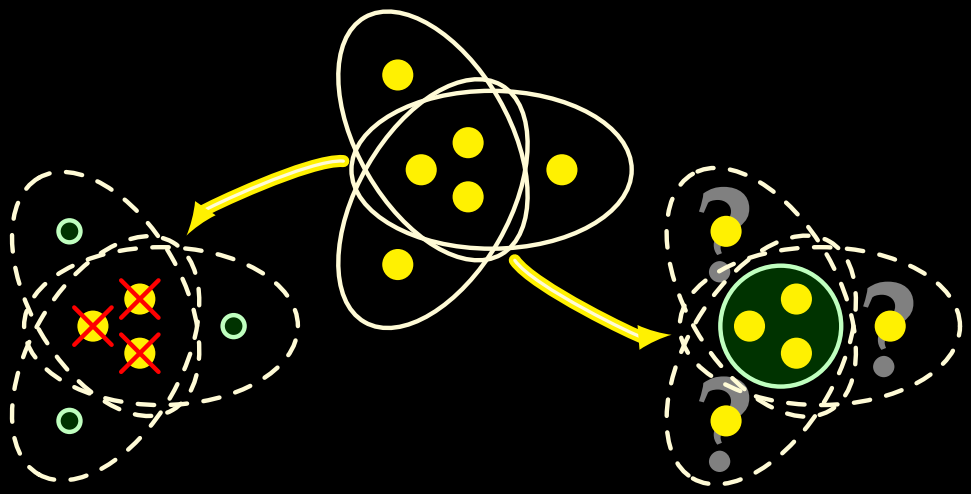

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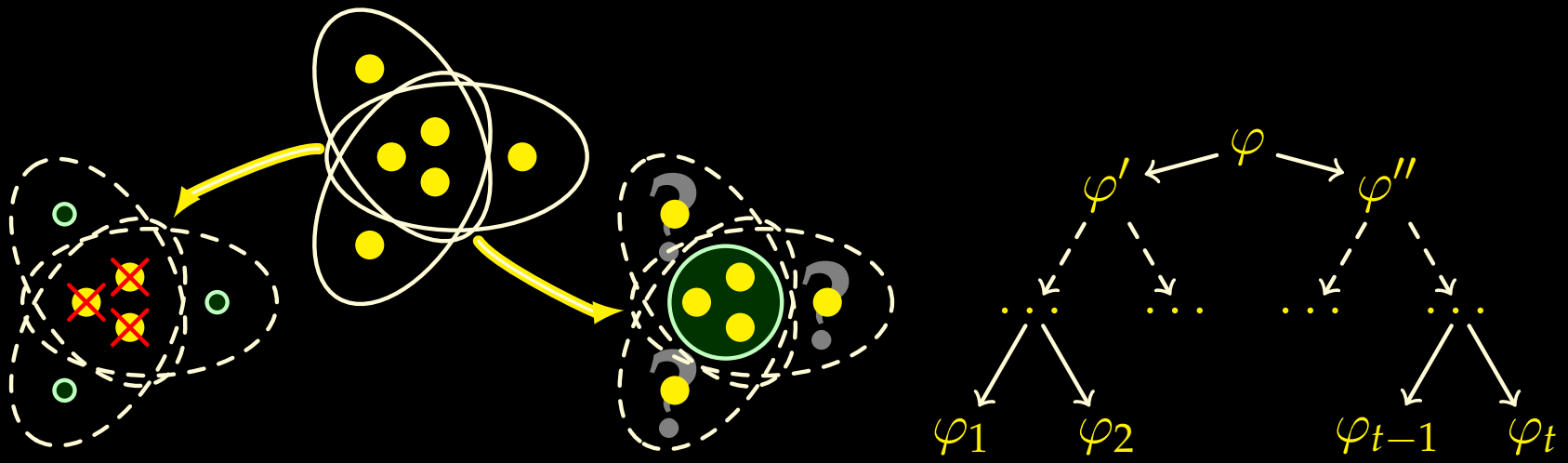

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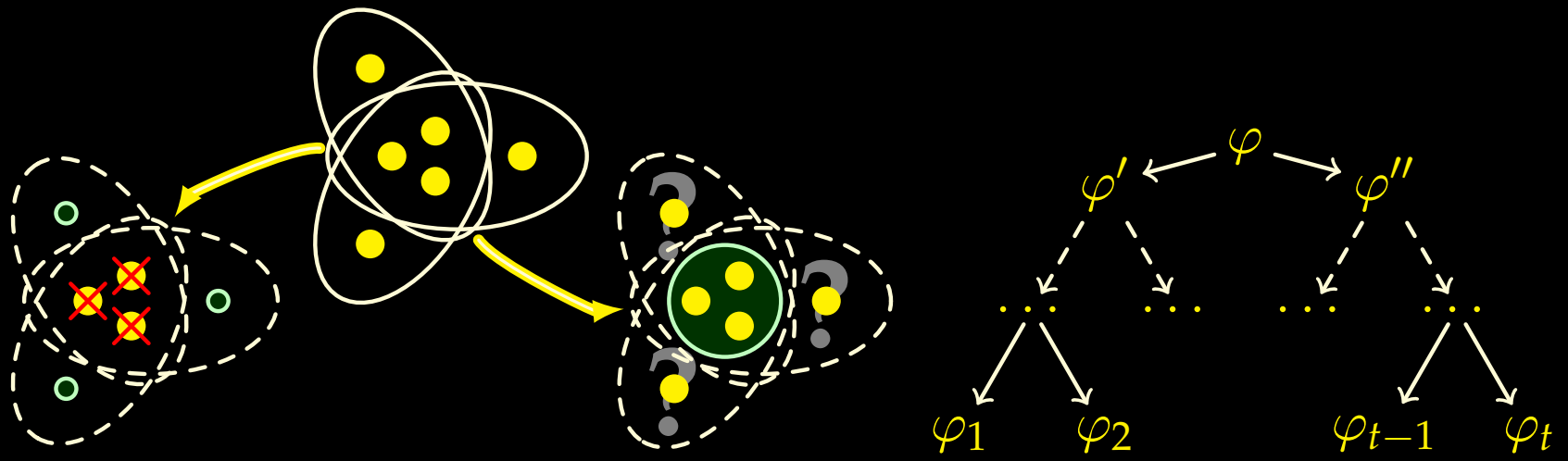

such that $\mathrm{sat}(\varphi) = \mathrm{sat}(\varphi_1) \dot\cup \ldots \dot\cup \mathrm{sat}(\varphi_t)$.

Next...

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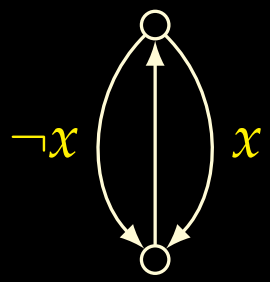

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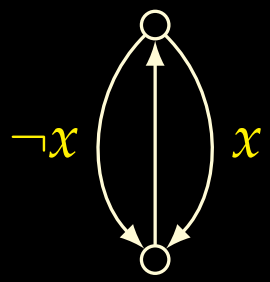

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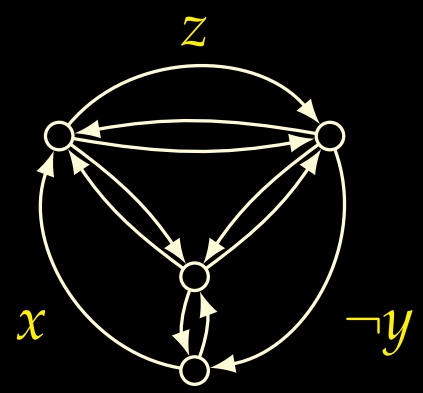

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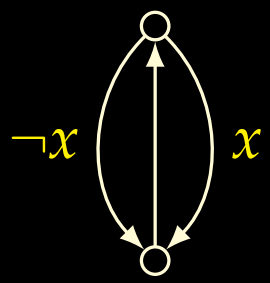

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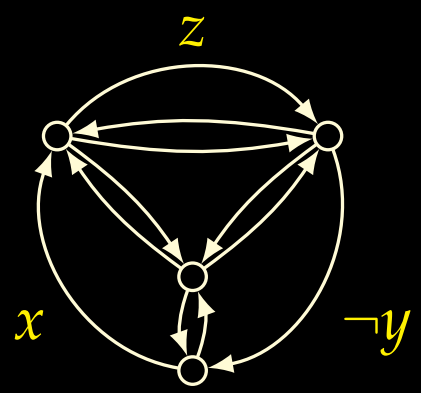

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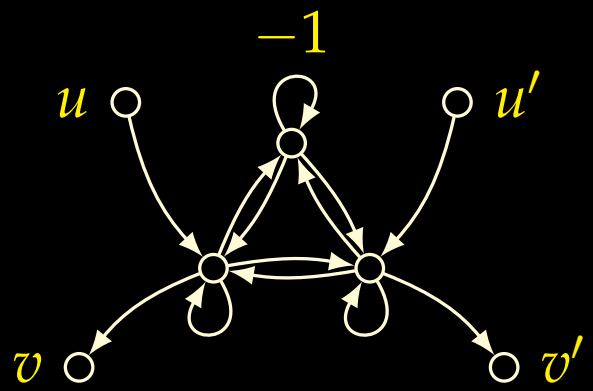

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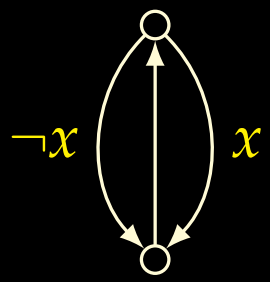

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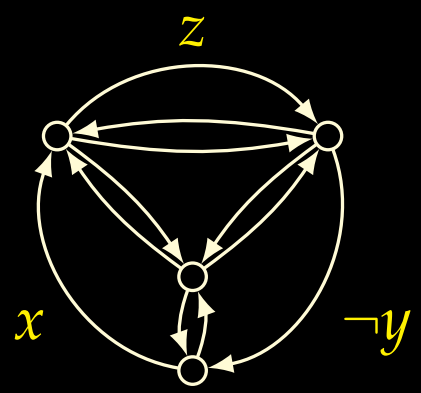

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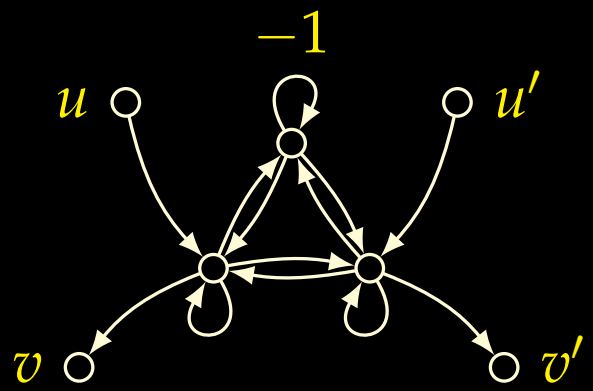

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"Isolated" formulas φ are still hard under ETH.

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U3SAT is not computable in time $2^{o(n)}$ unless ETH is false.
[Calabro, Impagliazzo, Kabanets, Paturi '08]

Permanent Results Review

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If Perm has $2^{o(d)}$-size formula exists, then ETH is false.

Permanent Results Review

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$$\mathrm{Perm}(A) = \sum_{S \subseteq [d]} (-1)^{d-|S|} \prod_{i=1}^{d} \sum_{j \in S} A_{i,j} \quad \text{with } 2^d \text{ terms.}$$

If Perm has $2^{o(d)}$-size formula exists, then ETH is false.

We don't know any barriers for 1.99^d-size formulas!

Strong Exponential Time Hypothesis

A tool for exploring tighter bounds

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...it seems difficult to show that 1.99^d-algorithms for the permanent imply that SETH fails

Part II:
The Tutte Polynomial

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of a graph $G = (V, E)$

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- the Ising model, the Potts model, ...

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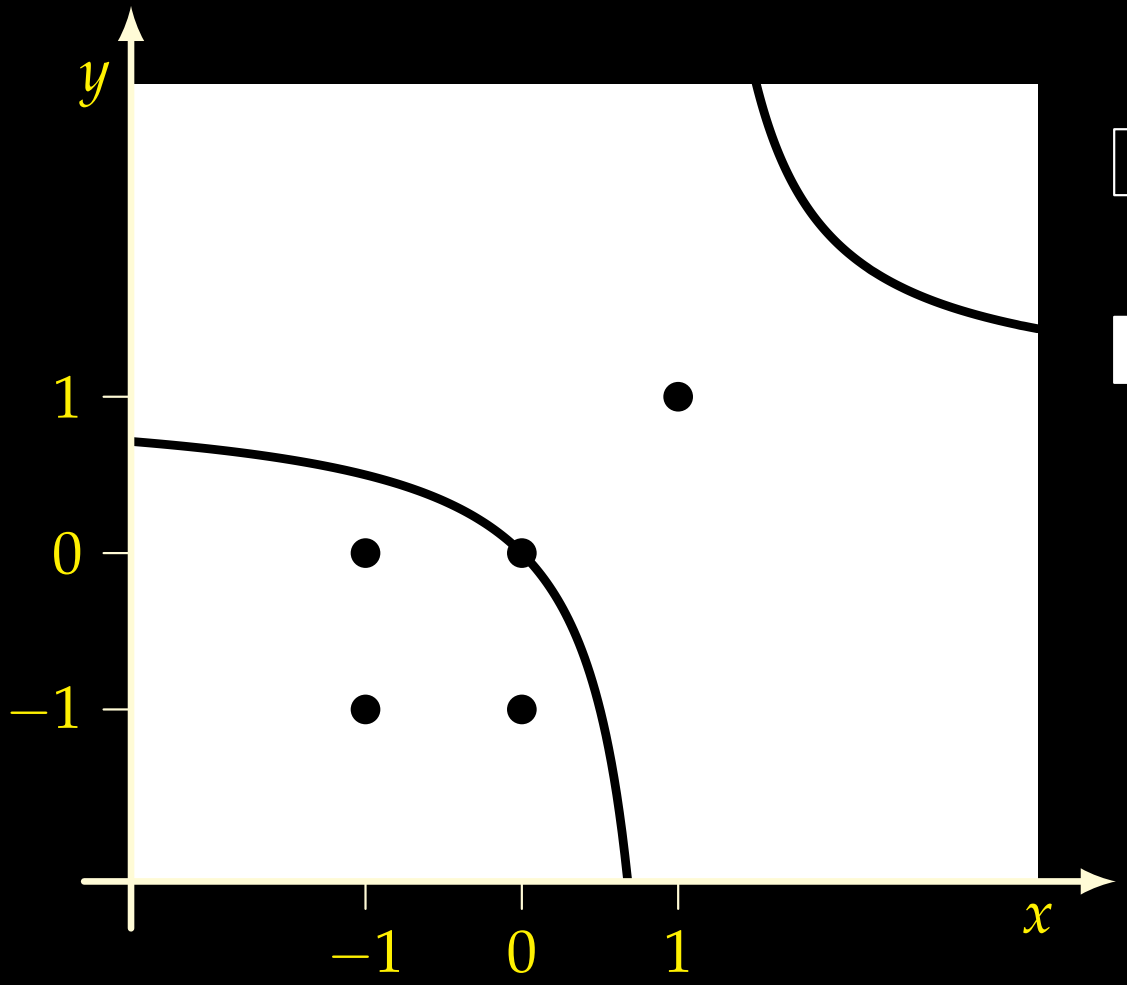
For fixed (x, y), compute the number $T(G; x, y)$ given G

The Tutte Plane under P $\neq$ NP

[Jaeger, Vertigan, Welsh '90]



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The Tutte Plane under ETH

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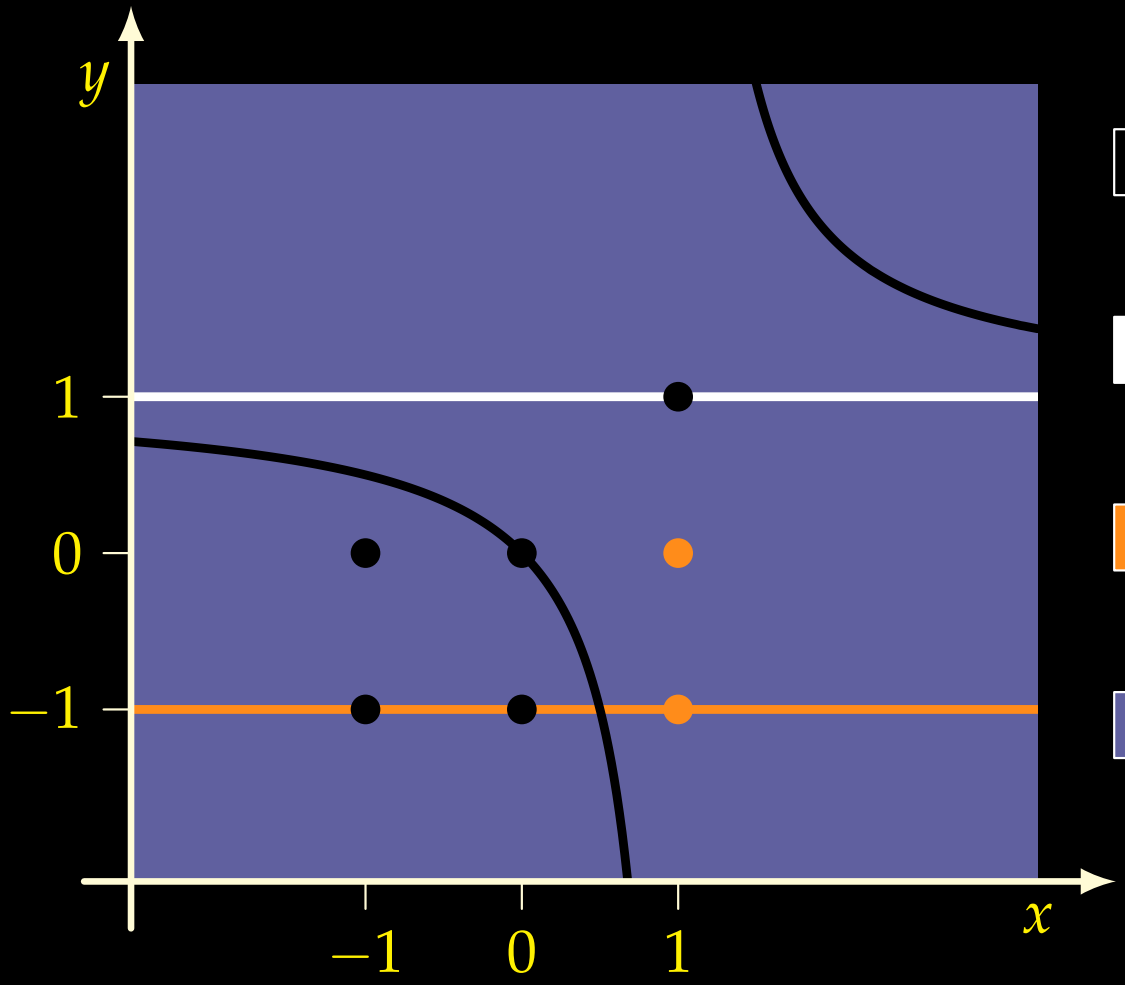

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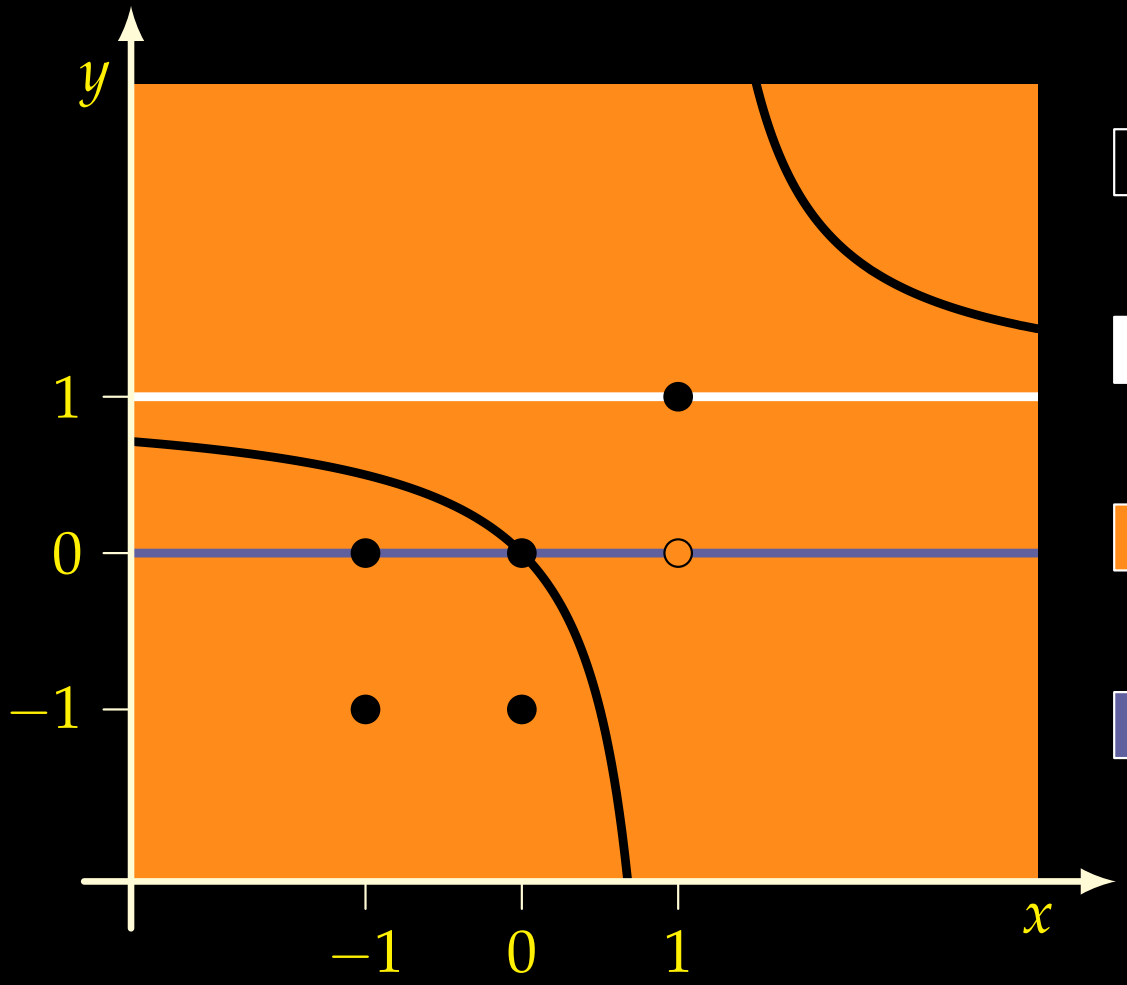

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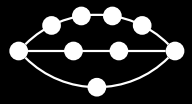
(1) Theta graphs: u  v

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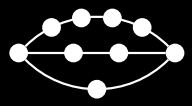
(1) Theta graphs: u  v

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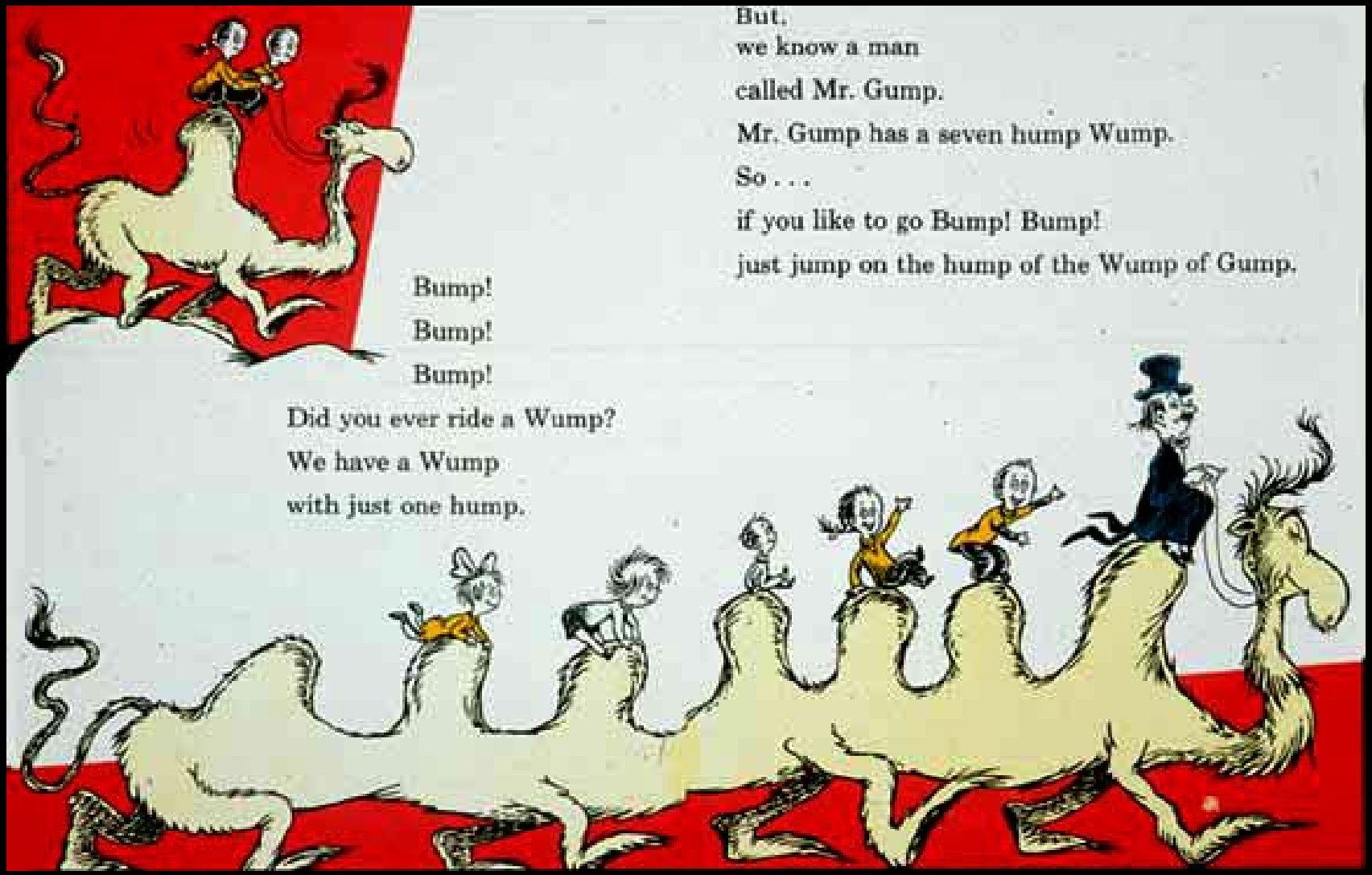

Dr. Seuss (1960): One Fish Two Fish Red Fish Blue Fish

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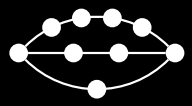
(1) Theta graphs: u  v

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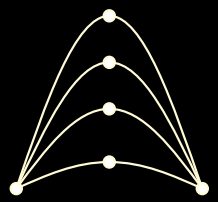

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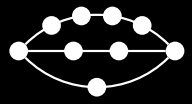
(1) Theta graphs: u  v

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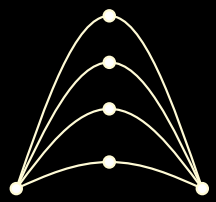

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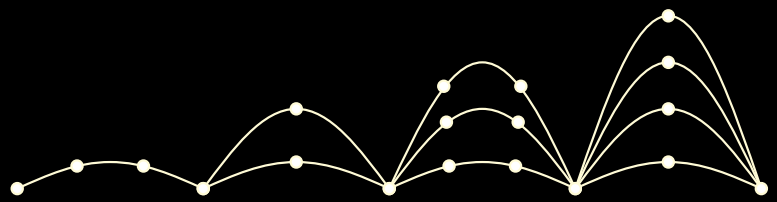
u  v

$\langle 3, 2, 3, 2 \rangle$-Wump

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- Does a Toda Theorem ($\mathrm{PH} \subseteq \mathrm{P}^{\#\mathrm{P}}$) hold for #ETH or #M[1]?

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