Combining Two Worlds: Parameterized Approximation for Vertex Cover

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Talk Overview

- Known results for VERTEX COVER:
  - Approximability
  - FPT
  - Parameterized Approximation
- Our factor 3/2 Approximation Algorithm
- Factor (2l+1)/(l+1) Approximation Algorithm

Vertex Cover

Problem name: VERTEX COVER (VC)

Given: A graph $G = (V,E)$

Parameter: a positive integer k

Output: Is there a *vertex cover* $C \subseteq V$ such that $|C| \leq k$?

Approximability

- There is a simple factor 2 approximation algorithm: select an edge and add both end vertices to the vertex cover; repeat until there are still edges remaining in the graph.

- The best known approximation factor for general graphs: *2-o(1)*

Approximability

Theorem: *(Dinur and Safra, 2002)*
Approximating vertex cover with constant factor less than 1.36 is NP-hard.

Previous best result: 1.1666 (Håstad 1997)

Approximability

<u>Theorem</u>: *(Subhash Khot and Oded Regev, 2008)*

Assuming that unique games conjecture holds, approximating vertex cover with constant factor less than 2 is NP-hard.

Approximability

Theorem: *(P. Berman and T. Fujito, 2005)*

There exists a polynomial time factor 7/6 approximation algorithm for minimum vertex cover for any graph with maximum degree 3.

Theorem: *(D. S. Hochbaum, 1983)*

There exists a polynomial time factor 3/2 approximation algorithm for minimum vertex cover for any graph with maximum degree 4.

Approximability

Theorem: *(M. M. Halldorsson and J. Radhakrishnan, 1997)*

For an arbitrary graph with average degree d_{avg}, there exist a

$$(4d_{avg}+1)/(2d_{avg}+3)$$

approximation algorithm for minimum vertex cover, assuming that there exists a minimum vertex cover that contains at least half of all vertices.

FPT

- General graphs: *O*(1.28^k)*
- Cubic graphs: *O*(1.194^k)*
- Graphs with bounded genus: O*(c$^{\sqrt{k}}$)
- Assuming that the Exponential Time Hypothesis holds, there can be no O*(c $^{o(k)}$) algorithm for general graphs *(Cai and Juedes 2003).*

Parameterized Approximation

Theorem: If there exists an exact exponential time algorithm for computing minimum vertex covers that runs in time $O^*(\gamma^n)$, then an approximation factor of $2-\rho$, for $\rho \in (0, 1]$ can be obtained with running time $O^*(\gamma^{\rho n})$ *(N. Bourgeois, B. Escoffier, and V. Th. Paschos, 2009).*

Parameterized Approximation

- Degree-0 Rule: Delete isolated vertices; they do not go into the cover.

- Degree-1 Rule: If v has a neighbour of degree one, put v into the cover and decrement the parameter.

- Degree-2 Rule: If N(y) = {x, z}, then remove both vertices x and y, together with their incident edges, and connect z to the remaining neighbours of x. Exactly one of the vertices x and y is in the cover and consequently we decrement the parameter k.

Approximation Preserving (AP) Degree-2 Rule.

We apply the AP Degree-2 Rule if there are no vertices of degree at most one but there is a vertex y of degree 2 with neighbours x and z such that there is no edge between x and z (if there is such an edge we apply the triangle rule described below).

We select one of the neighbours of y, say x, and an edge {x, u}, $u \neq y$, and add this to the set M. We then apply the Degree-1 Rule to vertex y and add its only remaining neighbour to the cover, that is, to the set C. Thus we add 3 vertices to the vertex cover and at least 2 of them must be included in any minimum cover. Therefore, we are within our 3/2 ratio.

Please note the difference between the Degree-2 Rule described in the previous section and the AP Degree-2 Rule described above. Importantly, while the Degree-0 Rule, Degree-1 Rule and AP Degree-2 Rule (that is, the Approximation Preserving Degree-2 Rule described above) never increase degrees of vertices (as we only remove vertices and their incident edges), such a degree increase may happen with the Degree-2 Rule presented in the previous Section, as well as with the AP Degree-3 Rule described below.

Approximation Preserving (AP) Degree-3 Rule.

If there are no vertices of degree at most 2, but there is at least one vertex v of degree 3, we then select a neighbour u of the vertex v and a neighbour $x \neq v$ of u and add {x, u} to M; in that way we have created a vertex of degree 2, namely v, and we apply the Degree-2 Rule described in the previous section to it; altogether we add 3 vertices to the vertex cover and some minimum cover must contain at least 2 of them, so we are within the 3/2 approximation ratio. This seems to provide a new way of dealing with vertices of degree three.

Triangles.

If there is a triangle we add all of its vertices to the vertex cover (again, at least 2 of them must be in every vertex cover so we are within the 3/2 ratio). This rule is actually well-known.

Algorithm 1

1. Apply approximation preserving reduction rules.
2. If $\Delta(G) < 5$, approximate in polynomial time.
3. Pick a vertex v of maximum degree and a vertex u at distance two from v so that $|N(u) \cup N(v)|$ is maximum.
4. If $N(u) \cup N(v) = N(v)$, the graph has at most $O(\Delta(G)^2)$ vertices and can be dealt with in constant time if $\Delta(G) < 10$.

   4a. If $\Delta(G)$ 10, we branch as described below in step 6.

Algorithm 1 (cont'd)

5. $|N(u) \setminus N(v)| = 1$ another special but easy case. (We distinguish 3 cases.)

   5a. There are no vertices at distance greater than 2 from v. Then we proceed as in case 4.

   5b. There is a single vertex w at distance 3 from v; then w is a cut vertex of the graph G — a contradiction.

   5c. There are at least 2 vertices at distance 3 from v and we proceed with branching as in case 6.

Algorithm 1 (cont'd)

6. If $|N(u) \setminus N(v)| \geq 2$, branch on two cases:

   6a. At least one of u, v is in any minimum cover respecting previous choices. Put u, v, x into the partial approximate cover, where $x \in N(u) \cap N(v)$.

   6b. None of u, v is is some minimum cover respecting previous choices. Put $(N(u) \cup N(v))$ into C. We have freed $f = |N(u) \cup N(v)|$ edges.

Factor (2l+1)/(l+1) approximation

l	1	2	3	4	10	20	100	200
d_l	1.13	1.08	1.06	1.05	1.02	1.01	1.002	1.001
c_l	1.09	1.04	1.024	1.017	1.0043	1.0014	1.00008	1.00002

Factor (2l+1)/(l+1) approximation