Open Problems Posed At WORKER 2010

1 Difficulty: Easy

Improved Linear (Vertex) Kernels for some problems on Planar Graphs

1. CONNECTED DOMINATING SET- best known is $120k$. Can we improve it to $\leq 60k$?
2. FEEDBACK VERTEX SET- best known is $112k$[BP08]. Can it be improved to $\leq 18k$?
3. VERTEX-DISJOINT CYCLE PACKING- best known is $909k - 717$[BPT08]. Can it be improved to $\leq 100k$?
4. DOMINATING SET- best known is $67k$[CFKX05]. Can we improve it to $\leq 35k$?
5. CONNECTED VERTEX COVER-best known is $14k$[GN07]. Can we improve it to $\leq 5k$?

Linear Kernel for Rooted Directed Max-leaf on Planar Graphs Does ROOTED PLANAR DIRECTED MAX-LEAF admit a linear kernel on planar graphs? (Harder) If so, what about on general graphs?

Polynomial Kernels for Subset Sum Parameterized by the Number of Numbers Does SUBSET SUM parameterized by the number of input numbers admit a polynomial kernel?

Consider the following parameterization of the SUBSET SUM problem:

SUBSET SUM

Given: Positive integers $s_1, s_2, \ldots, s_n$ and an integer t, all encoded in binary.
Question: Is there a subset of the input numbers which sums exactly to t?
Parameter: n

There is evidence to believe this problem has a polynomial kernel; Harnik and Naor show that it can be probabilistically compressed[HN06] . The Subset Sum problem parameterized by n is of similar flavor as the problem Exact Hitting Set parameterized by the universe size (i.e. given a universe U and a collection C of subsets of U, is there a subset U' of U which contains exactly 1 element from each set in C?), and this latter problem indeed has a polynomial kernel which can be obtained by observing that the constraints specify a set of linear equalities with one variable per universe element, and the rank of this system of linear equalities is at most $(|U| + 1)$ which implies that if there are more than $(|U| + 1)$ constraints (subsets in C) there must be one which we can safely delete without changing the answer to the problem.

When Subset sum is parameterized by $(k + d)$ where k is the number of input integers which must be summed to obtain the target value t, and d is the maximum number of bits in the binary representation of the input integers, there is no polynomial kernel unless the polynomial hierarchy collapses; this was proven by Dom, Lokshtanov and Saurabh[DLS09].

BiKernel for PLANAR INDEPENDENT DOMINATING SET

1. Is there a bikernel of size $O(k \log k)$ for PLANAR INDEPENDENT DOMINATING SET into any language?

2. Is there a bikernel of size $O(k)$ for PLANAR INDEPENDENT DOMINATING SET into any language?

3. (Harder) Demonstrate the *non-existence* of a linear bikernel for PLANAR INDEPENDENT DOMINATING SET into any language. (Note: a linear vertex kernel would imply a linear-in-bits kernel because of compact encodings of planar graphs.)

Polynomial Kernel for CONNECTED VERTEX COVER Find a suitable and non-trivial choice of a structural parameter r such that CONNECTED VERTEX COVER, when parameterized by *both* r and k, where k is the standard parameter (solution size), admits a polynomial kernel.

Polynomial Kernel for MULTI-WAY CUT

1. MULTI-WAY CUT (Edge Version)
   Given: A graph G, an integer k, and a subset T of the vertex set V (terminals)
   Question: Is there a subset of at most k edges after whose removal all terminal vertices lie in different components?
   Parameter: k
2. MULTI-WAY CUT (Vertex Version)
   Given: A graph G, an integer k, and a subset T of the vertex set V (terminals)
   Question: Is there a subset of at most k vertices after whose removal all terminal vertices lie in different components?
   Parameter: k

Both, the vertex and edge variants of the problem are known to be FPT[Mar04]. Do either (or both) admit a polynomial kernel?

Ecology Program and Well-Quasi Ordering BANDWIDTH, when parameterized by vertex cover number, turns out to be FPT[FLM+08] (note that graphs with small vertex covers could have large BANDWIDTH). The parameterized complexity of OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover number is still open.

OPTIMAL LINEAR ARRANGEMENT

Given: A graph $G = (V, E)$, an integer k
Question: Does there exist a layout L for V (a permutation $L(1), L(2), \ldots, L(n)$), such that the weight of the layout is at most k, where the weight of the layout $\mathbb{W}(G, L) = \sum_{(u,v)\in E} |L(u) - L(v)|$?
Parameter: t, where t is the vertex cover number of G.

Graphs of bounded vertex cover number are well-quasi ordered under induced subgraphs. Induced subgraphs in general is not a well-quasi order, but it is on graphs that have bounded vertex covers.

Ding's theorem[Din92] is a nice characterization of when a family of graphs is well-quasi ordered under the induced subgraph relation. Lay the foundation for the quest of universal anti-chains.

$P_{2/3}$-***packing*** Given l paths disjoint of length two (paths comprising of two edges on three vertices):

$$P_1, \ldots, P_l,$$

if we wish to check if there exists a collection of $(l+1)$ disjoint paths of length two, then it is known that among the $3l$ vertices involved in $P_1, \ldots, P_l$, $(2.5l)$ vertices can be "recycled"[FR09]. This means that we can use our knowledge of the previous iteration to somehow produce the next iteration.

Conjecture: We can recycle all $3l$ vertices.

2 Difficulty: Medium

A Faster Linear Kernel for VERTEX COVER. Find a linear vertex kernelization for VERTEX COVER which runs in time $O(m+n+k)$.

A Faster Quadratic Kernel for FEEDBACK VERTEX SET. The known quadratic kernel for FEEDBACK VERTEX SET runs in time $O((m+n)\sqrt{n})$[Tho09]. The question is to find a quadratic kernel that can be arrived at in time $O(n\alpha(n))$, where $\alpha(n)$ is the inverse-Ackerman function.

Decomposing Longest Path Given an instance (G, k) of k-PATH, is it possible to generate p instances,

$$(G_1, k_1), \ldots, (G_p, k_p),$$

such that:

1. for all $1 \leq i \leq p$, $k_i \leq k$ and $|G_i| = 2^{o(k)}$,
2. $p = 2^{o(k)}$, and,
3. (G, k) is a YES instance of k-PATH iff for some i, (G_i, k_i) is a YES instance of k-PATH.

The problem is in the context of undirected general graphs. Running an algorithm on a subexponential number of subexponential-sized instances could prove to be a significantly improved situation in practice, despite the non-existence of polynomial kernels.

3 Difficulty: Medium-Hard

Sub-quadratic Kernels For the following problems, find (vertex) kernels of size $O(k^{2-\varepsilon})$:

1. Edge Dominating Set
   Given: Graph $G = (V, E)$, integer k
   Question: Does G have an edge dominating set comprising at most k edges?
   Parameter: k
2. Tournament Feedback Vertex Set
   Given: Tournament $T = (V, E)$, integer k
   Question: Does T have a feedback vertex set of size at most k?
   Parameter: k
3. Minimum-Maximal Matching
   Given: Graph $G = (V, E)$, integer k
   Question: Does G have a maximal matching consisting of at most k edges?
   Parameter: k
4. Cluster Vertex Deletion
   Given: Graph $G = (V, E)$, integer k
   Question: Does there exist $X \subseteq V$ of size at most k such that deleting X from G results in a cluster graph (i.e., a graph where every connected component is a clique)?
   Parameter: k
5. 3- Hitting Set
   Given: Hypergraph $G = (V, E)$ where every hyperedge contains at most 3 vertices, integer k
   Question: Does there exist a set of at most k vertices which hits every hyperedge?
   Parameter: k

   Bonus Question: Establish that the problems 1-4 are equivalent in the sense that a sub-quadratic kernel for one of these problems would imply a sub-quadratic kernel for all the others. (Naturally, this will have to be demonstrated using an appropriate notion of equivalence; one that would preserve sub-qudraticity of the kernels!)

Approximate Crown Reductions If there is a vertex cover and an independent set of size larger than the vertex cover, then a normal crown can be found in polynomial time (Nemhausser-Trotter). If the independent set is at least *twice* as large as the vertex cover, then a double crown can be found in polynomial time .

We would like to apply the crown rule with marginal loss of information. What if you had something that was almost a crown, say, a head and a crown which didn't admit a full matching, but one that saturated nine vertices out of ten. What can be proven formally about a crown reduction that is applied in this situation? The question is to establish a reasonable notion of approximative crown reduction and discover how much can be "pulled back" from the application of a "lossy reduction rule". These rules are to be applied in less-than-ideal situations when full crowns cannot be found, and are potentially useful in practice.

Complexity of MaxLin2 AA

Given: given a system of m linear equations $\sum_{i \in I_j} z_i = b_i$ over $\mathbb{F}_2$, equation j has weight $w_j \in \mathbb{N}$.
Question: Does there exist an assignment $z^* \in \{0,1\}^n$ such that $W(z^*) \geq W/2 + k$, where $W(Z^*)$ is the total weight of the equations satisfied by z^* and $W = \sum_{i=1}^{m} w_i$.
Parameter: k

Is MAXLIN2 AA Fixed Parameter Tractable? If yes, does it have a polynomial kernel?

Polynomial Kernels for Independent Set or Clique The question of whether a graph G has either an independent set or a clique of size at least k is evidently FPT (from Ramsey theory; since the answer is always yes for graphs that are large enough, and anything smaller is an exponential kernel). Does this problem admit a polynomial kernel?

In fact, more can be said about large parties than the fact that they admit either a clique or an independent set of a certain size. The existence of 'relatively large' cliques or independent sets can also be shown. Consider a modified party scenario, where as people walk through the door they get a number. A member of the party thinks the group is large if the number of people in the group is larger than the number (s)he has been alloted. The famous Paris-Harrington theorem asserts that for every k there exists a party so large that there is a homogeneous set, a group of k mutual friends or strangers, such that at least one member in the group thinks that the party is large. This theorem is not provable in Peano arithmetic. The growth rate of the function is provable to be much larger than the Ackerman function.

Non-standard parameterization for subdivisions of small graphs Let for a graph G, $bb(G)$ be the minimum number of vertices of a graph H such that G can be obtained from H by subdividing edges. Is there a polynomial size kernel for the following problem, for each property ψ that can be expressed in Monadic Second Order Logic?

Given: Graph $G = (V, E)$, integer K
Question: Is there a set $W \subseteq V$ with $\psi(G, W)$ and $|W| \leq K$?
Parameter: $bb(G)$

It is not hard to see that $bb(G)$ can be computed in polynomial time. The question for maximumization is equivalent. A related question is when we take as parameter the number of vertices with degree at least three.

Counting Vertex Covers Does the following problem have a polynomial kernel?

Given: Graph $G = (V, E)$, vertex cover $W \subseteq V$, integer L
Question: Does G have at least L minimal vertex covers?
Parameter: $|W|$

Several variants of this question are also interesting.

Long Cycle parameterized by Feedback Vertex Set Does the following problem have a polynomial kernel?

Given: Graph $G = (V, E)$, feedback vertex set $W \subseteq V$, integer L
Question: Does G have a simple cycle of length at least L?
Parameter: $|W|$

Or-composition for Treewidth There is no polynomial kernel for TREEWIDTH, assuming that the AND-distillation conjecture holds, and TREEWIDTH is and-compositional: the treewidth of a graph equals the maximum treewidth of a connected component. But: is TREEWIDTH also or-compositional? I.e., is the following conjecture true?

Conjecture. There is no polynomial kernel for TREEWIDTH, unless $NP \subseteq coNP/poly$.

The same question can be posed for PATHWIDTH.

4 Difficulty: Hard

Existence or non-existence of polynomial kernels Demonstrate the existence or non-existence of polynomial kernels for the following problems.

1. ODD CYCLE TRANSVERSAL
   (Easier)PLANAR ODD CYCLE TRANSVERSAL
2. DIRECTED FEEDBACK VERTEX SET[1]

The AND-conjecture What are the implications of a problem having both an AND-composition and a polynomial kernel. A violation of a reasonable complexity-theoretic hypothesis would be nice to prove. On the other hand, the question could be to demonstrate a problem that admits both.

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